Journal of Statistical Physics, Vol. 59, Nos. 1/2, 1990

Statistical Mechanics of Probabilistic Cellular Automata

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Received May 16, 1989; revision received November 8, 1989

We investigate the behavior of discrete-time probabilistic cellular automata (PCA), which are Markov processes on spin configurations on a d-dimensional lattice, from a rigorous statistical mechanics point of view. In particular, we exploit, whenever possible, the correspondence between stationary measures on the space-time histories of PCAs on $\mathbb{Z}^d$ and translation-invariant Gibbs states for a related Hamiltonian on $\mathbb{Z}^{(d+1)}$. This leads to a simple large-deviation formula for the space-time histories of the PCA and a proof that in a high-temperature regime the stationary states of the PCA are Gibbsian. We also obtain results about entropy, fluctuations, and correlation inequalities, and demonstrate uniqueness of the invariant state and exponential decay of correlations in a high-noise regime. We discuss phase transitions in the low-noise (or low-temperature) regime and review Toom's proof of nonergodicity of a certain class of PCAs.

KEY WORDS: Probabilistic cellular automata; statistical mechanics; Gibbs measures.

1. INTRODUCTION

It has been recognized for a long time that there is an intimate relation between Probabilistic cellular automata (PCA), which include deterministic automata (CA) as special limits, in d dimensions and equilibrium statistical models (ESM) in $(d+1)$ dimensions, the extra dimension being the discrete time. This connection has been exploited by Domany[(1)] and others[(2–4)] to obtain information about the equilibrium properties of some

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0022-4715/90/0400-0117$06.00/0

$(d+1)$-dimensional spin systems from knowledge of specially constructed d-dimensional PCAs. There has been, however, only little general study of te connection from a mathematically rigorous point of view.[5,6] It is the purpose of this article to continue this general study. We include, for pedagogical reasons, also a review of some previous work (see refs. 7 and 8 for earlier reviews of PCAs). In particular, we give the proof (we hope in simpler language) of a very beautiful theorem by Toom[9] proving non-ergodic behavior for certain PCAs. We hope that this will increase awareness of the result in the statistical physics community.

1.1. PCA Formalism

We consider PCAs which describe the stochastic discrete-time evolution of Ising spin variables on regular lattices; for the moment we discuss only the lattice $\mathbb{Z}^d$. We denote the value of the spin at site $i \in \mathbb{Z}^d$ at time $n \in \mathbb{Z}$ by $\sigma_{n,i} = \pm 1$, and write $\underline{\sigma}_n = \{\sigma_{n,i}\}_{i \in \mathbb{Z}^d}$ for the configuration at time n; we will occasionally let $\underline{\eta}$ denote a generic configuration on $\mathbb{Z}^d$. The PCA evolves by simultaneous independent updating of spins. That is, the spin configuration $\underline{\sigma}_{n-1}$ determines the probabilities $p_i(\sigma_{n,i} \mid \underline{\sigma}_{n-1})$ of the spin values at each site i at time n, and the conditional probability distribution of the corresponding $\underline{\sigma}_n$ is a product measure which we denote by $P(d\underline{\sigma}_n \mid \underline{\sigma}_{n-1})$. The conditional probability of the configuration $\underline{\sigma}_n$ is thus given formally by

$$\prod_{i \in \mathbb{Z}^d} p_i(\sigma_{n,i} \mid \underline{\sigma}_{n-1}) \tag{1.1}$$

The transition probabilities satisfy the normalization condition

$$\sum_{\sigma_{n,i} = \pm 1} p_i(\sigma_{n,i} \mid \underline{\sigma}_{n-1}) = 1 \tag{1.2}$$

which is automatically taken into account by writing

$$p_i(\sigma_{n,i} \mid \underline{\sigma}_{n-1}) = \tfrac{1}{2}[1 + \sigma_{n,i} h_i(\underline{\sigma}_{n-1})] \tag{1.3}$$

with $|h_i(\underline{\sigma}_{n-1})| \leqslant 1$. We assume that h_i is translation invariant and of finite range, so that it may conveniently be written in the form of a finite sum over finite subsets of $\mathbb{Z}^d$:

$$h_i(\underline{\eta}) = \sum_A r_A \eta_{A+i} \tag{1.4}$$

where $\eta_B = \prod_{j \in B} \eta_j$ and $A + i$ denotes the translation of A by the lattice vector i. The r_A in (1.4) are real numbers and may be viewed as the parameters which govern the behavior of the PCA.

If

$$h_i(\underline{\eta}) = M_i(\underline{\eta}) \tag{1.5}$$

where M_i is a local function taking only the values $+1$ and -1, then the PCA is a (deterministic) CA. Since the functions $h(\underline{\eta})$ which depend on k spins and satisfy $|h(\underline{\eta})| \leqslant 1$ form a cube in $\mathbb{R}^{2^k}$ whose corners correspond to the CA (1.5), every h is a convex combination of such M:

$$h_i(\underline{\eta}) = \sum_{\alpha=1}^{m} q_\alpha M_{\alpha,i}(\underline{\eta}), \qquad \sum_{\alpha=1}^{m} q_\alpha = 1, \qquad q_\alpha \geqslant 0 \tag{1.6}$$

The evolution of the PCA (1.6) may be pictured as follows: at each site a choice of deterministic rule is made, independently of other sites and of the input $\underline{\sigma}_{n-1}$, with rule M_α chosen with probability q_α, so that

$$\sigma_{n,i} = M_{\alpha,i}(\underline{\sigma}_{n-1}) \qquad \text{with probability } q_\alpha \tag{1.7}$$

Applications of this representation are discussed in Section 6.

Equations (1.1)–(1.4) define a discrete-time *Markov process* on the space of spin configurations on $\mathbb{Z}^d$. Given a measure ρ_{n-1} on the configurations $\underline{\sigma}_{n-1}$, (1.1) defines a probability measure $\rho_n = \rho_{n-1}P$ on the configurations σ_n:

$$\rho_n(d\underline{\sigma}_n) = \int \rho_{n-1}(d\underline{\sigma}_{n-1})\, P(d\underline{\sigma}_n \mid \underline{\sigma}_{n-1}) \tag{1.8}$$

We say that a measure ν is *stationary* or *time invariant* if $\nu = \nu P$; it is *periodic*, with period k, if $\nu P^k = \nu$. The stationary measures for a PCA form a nonempty convex set.

1.2. From PCA to ESM...

It is natural to consider $\{\underline{\sigma}_n\}_{n\in\mathbb{Z}}$ as defining a spin configuration $\underline{\sigma}$ on the space-rime lattice $\mathbb{Z}^{d+1}$; we will write $x = (n, i)$ for a typical site in this lattice and let $\mathbb{Z}_n^d$ denote the d-dimensional layer corresponding to time n. By the usual convention of cellular automata, we will visualize the time axis in $\mathbb{Z}^{d+1}$ as vertical and oriented so that the past is at the top and the future at the bottom. If ρ is a measure on the state space of the PCA and we "start" the time evolution with measure ρ on the layer $\mathbb{Z}^d_{-N}$, then the Markov transition rates (1.1) define a measure μ_ρ^{-N} on the set of configurations on $\bigcup_{n\geqslant -N} \mathbb{Z}_n^d$. When $\rho = \nu$ is taken to be stationary for the time evolution, the $N \to \infty$ limit produces a measure μ_ν on the set of space-time configurations $\{\underline{\sigma}\}$, which is translation invariant in the time direction and whose projection on any $\mathbb{Z}_n^d$ is just ν. Similar conclusions hold when ν is periodic.

It is a simple observation that if the transition probabilities $p_i(\sigma_{n,i} | \underline{\sigma}_{n-1})$ are all strictly positive, i.e., if

$$|h_0(\underline{\eta})| < 1 \qquad \text{for all } \underline{\eta} \tag{1.9}$$

then μ_ν is a Gibbs measure for the Hamiltonian

$$\mathscr{H}(\sigma) = \sum_{x = (n, i) \in \mathbb{Z}^{d+1}} H_x(\sigma_x, \underline{\sigma}_{n-1}) \tag{1.10}$$

Here, for $x = (n, i)$ the single-site energy is defined by

$$\exp[-H_x(\sigma_x, \underline{\sigma}_{n-1})] \equiv p_i(\sigma_{n,i} | \underline{\sigma}_{n-1}) \tag{1.11}$$

Equation (1.2) now becomes

$$\sum_{\sigma_x = \pm 1} \exp[-H_x(\sigma_x, \underline{\sigma}_{n-1})] = 1 \tag{1.12}$$

The reciprocal temperature β which usually multiplies the energy in the exponent of (1.11) has been absorbed into H_x.

There are various procedures to define finite-volume Gibbs states which, in the thermodynamic limit, yield the space-time measure μ_ν of the PCA as an infinite-volume Gibbs state with respect to (1.10). Essentially, one must take the finite-volume Gibbs state of (1.10) in special domains and impose boundary conditions determined by ν on the top, i.e., on the boundary corresponding to the past, and appropriate free boundary conditions on the sides and on the bottom (i.e., the future). For more details, see ref. 6. Note that any measure μ_ρ^{-N} is a Gibbs measure on the semi-infinite $(d+1)$-dimensional domain $\bigcup_{n > -N} \mathbb{Z}_n^d$ with boundary condition ρ on the top.

1.3. ... and Back

In the general study of ESM corresponding to some interaction Hamiltonian one is interested in *all* infinite-volume Gibbs measures, that is, in all measures which satisfy the DLR equations.[10] These are obtained as limits of finite-volume Gibbs states defined with arbitrary boundary conditions. The main result in ref. 6 is a characterization of all the translation-invariant or periodic Gibbs measures for those ESM which arise from the PCA → ESM construction, that is, for ESM with Hamiltonian (1.10) satisfying the constraint (1.12): if μ is such a Gibbs measure, then $\mu = \mu_\nu$ for some stationary (or time-periodic) measure ν for the PCA. Consequently, the existence of more than one periodic phase for the ESM implies the existence of more than one stationary ν for the PCA, obtained by projection of μ onto $\mathbb{Z}^d$. Whether a similar result holds for nonperiodic (in space or time) Gibbs measures remains an open question.

An important consequence of (1.12) is that the finite-volume free energy for the ESM is identically zero for the special domains and boundary conditions mentioned above[6] or for rectangular regions with periodic boundary conditions in space, arbitrary initial conditions at the top, and free boundary conditions at the bottom. Moreover, for any boundary condition, the free energy is of the order of the size of the boundary region. Hence the infinite-volume free energy density, which is independent of boundary conditions, is identically zero. In particular, it is analytic in the parameters r_A of (1.4), even when, as we discuss below, there is a phase transition in the sense of ESM. The same analyticity will hold for the dependence of the free energy on the interaction coefficients entering the Hamiltonian (1.10), whose relation to the r_A will be studied shortly. In certain cases this analyticity may be shown to hold separately in the entropy and energy densities, even when there is a phase transition as the parameters change; see (1.30)–(1.32).

1.4. Phase Transitions

We shall refer to any change in some parameter r_A which changes the number of stationary or periodic measures ν for the PCA as a *phase transition*. The existence of more than one stationary or periodic ν, which corresponds to phase coexistence in the ESM, is to be contrasted to the case of *ergodic* behavior for the PCA when, for any initial state ρ, ρP^n converges as $n \to \infty$ to a unique invariant measure ν. One way to ensure nonergodicity in the PCA, realized in a construction due to Domany,[1] is to consider a PCA with dimension $d = 2$ for which all Gibbs states for the Ising model with nearest-neighbor ferromagnetic interactions are stationary. This Ising model is known to have a phase transition as the interaction strength J is varied. Hence, for $J > J_c$ there will be two stationary states ν_+ and ν_- for the PCA and two phases μ_{ν_+} and μ_{ν_-} for the three-dimensional ESM, corresponding to positive and negative spontaneous magnetization. In Section 5 we discuss this model further, as well as a class of models due to Toom,[9] also with $d = 2$, in which the two-dimensional stationary measures are not explicitly known, but for which Toom was able to prove nonergodicity using (in the ESM language) a rather unconventional Peierls argument. Whether such phase transitions can also occur in $d=1$ PCAs, corresponding to ESM in two dimensions, is an interesting question: see the discussion of the *positive rates conjecture* in ref. 13, and also ref. 14. Some of the issues are illustrated in the following example.

Example 1.1. *Majority rule in one dimension.* A spin at time n takes the value of the majority of itself and its two neighbors at time $n-1$ with probability $1-\varepsilon$; in the notation of (1.3),

$$h_i(\underline{\eta}) = (1-2\varepsilon)\,\mathrm{sgn}(\eta_{i-1}+\eta_i+\eta_{i+1}) \tag{1.13}$$

with $\mathrm{sgn}(\lambda)$ the sign of λ. The single-site Hamiltonian corresponding to (1.13) is

$$\begin{aligned} H_0(\sigma_0, \underline{\sigma}_{-1}) = {} & -\beta\sigma_0(\sigma_{-1,-1}+\sigma_{-1,0}+\sigma_{-1,1}) \\ & + \beta\sigma_0\sigma_{-1,-1}\sigma_{-1,0}\sigma_{-1,1} + \log(2\cosh 2\beta) \end{aligned} \tag{1.14}$$

where $\beta = (1/4)\log[(1-\varepsilon)/\varepsilon]$. The Hamiltonian H_0 contains ferromagnetic two-body and antiferromagnetic four-body interactions. Due to the balance between these two, there is an infinite number of ground states corresponding to trajectories of the (deterministic) cellular automata obtained by setting $\varepsilon=0$ or $\varepsilon=1$. These properties are in fact typical for equilibrium systems on the *disorder line* in parameter space.[11,12] The case $\varepsilon=1/2$ gives $\beta=0$, which corresponds to infinite temperature in the ESM. It is easy to show that there is a unique Gibbs measure for (1.14) if β is sufficiently small and, correspondingly, ergodic behavior of the PCA for ε close to 1/2 ($1/4<\varepsilon<3/4$ suffices, by the results of Section 5). Gray[15] showed that the PCA is ergodic for sufficiently small ε, and recently[16] has extended the result to prove that this one-dimensional PCA has a unique invariant measure for all ε, $0<\varepsilon<1$. This example is discussed further at several points in Section 6.

1.5. Structural Relation of PCA with ESM

The normalization (1.12) imposes strong constraints on the interactions of the $(d+1)$-dimensional ESM arising from PCA. Also, it is clear that due to the special role of the past in a PCA, the interactions will in general be highly anisotropic. To obtain explicit relations, we rewrite (1.11) using (1.3) as

$$H_0(\sigma_0, \underline{\eta}) \equiv -\sum J_R\sigma_R = \sigma_0 Q_0(\underline{\eta}) + \log[2\cosh Q_0(\underline{\eta})] \tag{1.15}$$

with

$$Q_0(\underline{\eta}) = \frac{1}{2} \log \left[\frac{1 - h_0(\underline{\eta})}{1 + h_0(\underline{\eta})} \right] \tag{1.16}$$

that is,

$$h_0(\underline{\eta}) = -\tanh Q_0(\underline{\eta}) \tag{1.17}$$

Rewriting (1.3) as

$$h_0(\underline{\eta}) = \sum_{\alpha=1}^{m} \tilde{r}_\alpha \eta_{A_\alpha} \tag{1.18}$$

with $\tilde{r}_\alpha \equiv r_{A_\alpha}$, we find from (1.11)

$$\begin{aligned} H_0(\sigma_0, \underline{\eta}) &= -\log\{[1 + \sigma_0 h_0(\underline{\eta})]/2\} \\ &= -\sigma_0 \sum_{\substack{\gamma \subset \{1,\dots,m\} \\ |\gamma| \text{ odd}}} K_\gamma \prod_{\alpha \in \gamma} \eta_{A_\alpha} \\ &\quad + \sum_{\substack{\gamma \subset \{1,\dots,m\} \\ |\gamma| \text{ even}}} K_\gamma \prod_{\alpha \in \gamma} \eta_{A_\alpha} + \log 2 \end{aligned} \tag{1.19}$$

where

$$K_\gamma = \sum_{k_1,\dots,k_m=0}^{\infty}{}' \prod_{\alpha=1}^{m} \frac{\tilde{r}_\alpha^{k_\alpha}}{k_\alpha!} \left[\sum_{\alpha=1}^{m} k_\alpha - 1 \right]! \tag{1.20}$$

and in the primed sum the k_α are not all zero and k_α is odd (respectively even) if $\alpha \in \gamma$ (respectively $\alpha \notin \gamma$). The J_R are now easy to compute. We also note here that each "bond" entering into the interaction $\log[2 \cosh Q_0(\underline{\eta})]$, that is, each set B such that η_B appears in the second sum in (1.19), is a symmetric difference of an even number of bonds in $Q_0(\underline{\eta})$.

Conversely, given any translation-invariant Hamiltonian on $\mathbb{Z}^{d+1}$ which can be written as a sum of H_x's, with H_0 of the form (1.15), there is a corresponding PCA defined by (1.17). For example, if we take $Q_0(\underline{\eta}) = -J\eta_A$, then

$$h_0(\underline{\eta}) = (\tanh J)\, \eta_A \tag{1.21}$$

while for $Q_0(\underline{\eta}) = -J_1 \eta_{A_1} - J_2 \eta_{A_2}$, we have

$$h_0(\underline{\eta}) = \tilde{r}_1 \eta_{A_1} + \tilde{r}_2 \eta_{A_2} \tag{1.22}$$

with

$$\tilde{r}_\alpha = \frac{\sinh 2J_\alpha}{\cosh 2J_1 + \cosh 2J_2}, \qquad \alpha = 1, 2 \tag{1.23}$$

The situation gets of course more complicated as the number of sets A increases. It is, however, useful to note that the relation between the coefficients of the η_{A_α} in h_0 and Q_0 given in (1.19)–(1.23) remains valid when the η_{A_α} there are replaced by functions $M_\alpha(\underline{\eta})$ such that $|M_{A_\alpha}| = 1$, as in (1.6) or (1.13).

1.6. Entropy

We now consider the problem of defining some measure of the noise or randomness in a PCA. If we are given some spin configuration $\underline{\eta}$ on $\mathbb{Z}^d$, then the natural measure of the randomness in the conditional distribution $p_0(\sigma_0 \mid \underline{\eta})$ for the spin at site 0 is the conditional entropy

$$s_0(\underline{\eta}) \equiv -\sum_{\sigma_0 = \pm 1} p_0(\sigma_0 \mid \underline{\eta}) \log p_0(\sigma_0 \mid \underline{\eta}) \tag{1.24}$$

We define the *noise* associated with the PCA as the maximum possible value of this conditional entropy:

$$\mathcal{N} \equiv \max_{\underline{\eta}} s_0(\underline{\eta}) \tag{1.25}$$

This definition has the attractive feature that the PCA noise is an upper bound for the entropy per site $\hat{s}(\mu_v)$ for any translation-invariant or periodic state μ_v of the ESM, which is in fact the same as the Kolmogorov–Sinai space-time entropy for the PCA process.[17] This entropy was shown in ref. 6 to be given by $\langle s_0 \rangle_v$, which from (1.25) clearly satisfies

$$\hat{s}(\mu_v) = \langle s_0 \rangle_v \leqslant \mathcal{N} \tag{1.26}$$

The definition (1.25) can be generalized immediately to PCA in which the spins can take on more than two values.

In our current case in which $\sigma = \pm 1$ it is natural to think in terms of the functions $h_i(\underline{\eta})$ defined in (1.3), which characterize the evolution of the PCA. Then

$$\mathcal{N} = -e \log e - (1-e) \log(1-e) \tag{1.27}$$

where $e = e(h) = \frac{1}{2} \max_{\underline{\eta}} (1 - |h(\underline{\eta})|)$. The simplest case is that of a (deterministic) CA (1.5), for which the noise is of course zero. In other simple cases the noise is a monotonic function of a single parameter. Frequently used rules of this type are

$$h_i(\underline{\eta}) = (1-2\varepsilon)\, M_i(\underline{\eta}), \qquad 0 \leqslant \varepsilon \leqslant 1/2 \tag{1.28}$$

with M_i as above, of which the majority rule (1.13) is an example, and a rule (sometimes called *thermal dynamics*[18]) analogous to conventional Monte Carlo updating:

$$h_i(\underline{\eta}) = \tanh\left(\beta \sum_{j \in U+i} \eta_j + b\right) \tag{1.29}$$

with $U \subset \mathbb{Z}^d$ some finite set.

We make several additional remarks. First, $s(\underline{\eta})$ is independent of $\underline{\eta}$ for PCA of the type (1.28), so that the ESM entropy per site is

$$\hat{s}(\mu_v) = s(\underline{\eta}) = -\varepsilon \log \varepsilon - (1-\varepsilon)\log(1-\varepsilon) \tag{1.30}$$

It is easy to see that

$$\langle \sigma_0 M_0(\underline{\sigma}_{-1}) \rangle = 1 - 2\varepsilon \tag{1.31}$$

and from (1.30) and (1.31) to rederive the vanishing of the infinite-volume free energy (the pressure) for these models:

$$\hat{s}(\mu_v) - \langle H_0(\sigma_0 \mid \underline{\sigma}_{-1}) \rangle_{\mu_v} = 0 \tag{1.32}$$

Second, by the concavity of $f(t) = -t \log t$ we have

$$\begin{aligned} \hat{s}(\mu_v) &\leqslant f(\langle p(1 \mid \underline{\eta}) \rangle_v) + f(\langle p(-1 \mid \underline{\eta}) \rangle_v) \\ &= f((1+m)/2) + f((1-m)/2) \end{aligned} \tag{1.33}$$

where $m = \langle \sigma_0 \rangle_{\mu_v}$. Finally, it follows from (1.26) that the "zero-temperature" state obtained as a limit of Gibbs states of the ESM when the PCA noise goes to zero has zero entropy per site, i.e., that the degeneracy of the ground state of the ESM cannot be too large.

1.7. Outline of the Paper

In Section 2 we make use of the PCA–ESM correspondence to apply the theory of fluctuations and of large deviations for Gibbs states to the time evolution of the PCA. This yields simple Gibbs measures, generally not of the PCA type, for the probability distributions of configurations in a space-time cube characterized by having a large fluctuation in the magnetization density. Section 3 is devoted to the review and extension of inequalities which hold for certain special PCAs: those having *positive dynamics* and those which are *attractive.* In Section 4 we give a specification of a "high-noise" regime and a characterization of the unique invariant measure which exists there: it is shown to be a Gibbs measure for a d-dimensional Hamiltonian and to have exponentially decaying correlations. In Section 5 we discuss phase transitions and present the models of Domany, which involves updating the lattice in several steps, and Toom. The Toom model phase diagram is discussed, following Bennett and Grinstein.[19] We also describe there a new mean field approximation. Section 6 is devoted to various remarks and some open problems. The proof of Toom's theorem is given in the Appendix.

It should be noted finally that we do not consider in this paper the types of PCAs which are used to model hydrodynamics (for an introduction to such models, with further references, see ref. 20). These PCA evolve by particle exchanges and their special behavior depends on the existence of corresponding *a priori* conservation laws, so that in particular (1.9) is violated; the conservation laws introduce constraints on the permissible configurations which make the space-time measures behave in a quite different way from those considered here. It is posssible, however, that such models may be profitably studied as limits of PCAs to which the PCA–ESM correspondence applies.

2. FLUCTUATIONS

The PCA–ESM correspondence has a straightforward application to the problem of space-time fluctuations in the stationary Markov process. We can apply here directly the well-developed theory of fluctuations—both typical[21,22] and large deviations[23,24]—of Gibbs states.

Let us start with the normal fluctuations of μ_ν, where ν is a translation-invariant stationary state of the PCA with average magnetization $m = \langle \sigma_0 \rangle$. We then have, whenever the truncated pair correlation

$$\langle \sigma_0 \sigma_x \rangle - \langle \sigma_0 \rangle \langle \sigma_x \rangle = \langle \sigma_0 \sigma_x \rangle - m^2 \tag{2.1}$$

defined with respect to μ_ν, is summable, that the mean square fluctuations in a box $\Lambda \subset \mathbb{Z}^{d+1}$, divided by $|\Lambda|$,

$$\chi_\Lambda \equiv \frac{1}{|\Lambda|} \left\langle \left[\sum_{x \in \Lambda} (\sigma_x - m) \right]^2 \right\rangle \tag{2.2}$$

tends, as $\Lambda \nearrow \mathbb{Z}^{d+1}$, to a well-defined limit,

$$\chi \equiv \sum_{x \in \mathbb{Z}^{d+1}} [\langle \sigma_0 \sigma_x \rangle - m^2] = \lim_{\Lambda} \chi_\Lambda \tag{2.3}$$

Under the same conditions when μ_v satisfies the FKG inequalities and under somewhat stronger conditions otherwise,[21,22] we also have that for $r \in \mathbb{Z}^{d+1}$ the field

$$\xi_L(r) \equiv \frac{1}{L^{(d+1)/2}} \sum_{x \in \Gamma_{Lr}} [\sigma_x - m] \tag{2.4}$$

where Γ_{Lr} is a cube of side L centered at Lr, converges (in distribution) to a free Gaussian field $\xi(r)$ with mean zero and variance χ.

Similar results hold for fluctuations in other local variables, e.g., if $F_A(\underline{\sigma})$ is a function which depends on σ_A, A a fixed set contained in a finite cube with center at 0, and $A+x$ its translate by x, then

$$\begin{aligned} \chi_\Lambda(F) &\equiv \frac{1}{|\Lambda|} \left\langle \left[\sum_{x \in \Lambda} F_{A+x} - \langle F_A \rangle \right]^2 \right\rangle \xrightarrow[\Lambda \nearrow \mathbb{Z}^{d+1}]{} \chi(F) \\ &\equiv \sum_{x \in \mathbb{Z}^{d+1}} [\langle F_{A+x} F_A \rangle - \langle F_A \rangle^2] \end{aligned} \tag{2.5}$$

The convergence to a Gaussian free field now proceeds as before.

2.1. Large Deviations

Let us consider again the space-time measure μ_v in which $\langle \sigma_0 \rangle = m$, which we assume to be translation invariant. We may now ask for the probability to find in a box $\Lambda \subset \mathbb{Z}^{d+1}$ a magnetization

$$M_\Lambda(\underline{\sigma}) = \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \sigma_x \tag{2.6}$$

which is larger than some $M \in [-1, 1]$, or, more generally, takes on a value in some interval (a, b). If $M > m$, or if $m \notin (a, b)$, then these events are very unlikely and the probability of their occurrence will go to zero as the box Λ grows. The PCA–ESM correspondence makes available the well-established large-deviation theory for ESM[23,24] to study the history of the stationary PCA. We define (free energy) functions

$$f_\Lambda(\theta) \equiv \frac{1}{|\Lambda|} \log \langle \exp[\theta\, |\Lambda|\, M_\Lambda(\underline{\sigma})] \rangle \tag{2.7}$$

and let

$$f(\theta) \equiv \lim_{\Lambda \uparrow \mathbb{Z}^{d+1}} f_\Lambda(\theta) \tag{2.8}$$

$f(\theta)$ will exist for all $\theta \in \mathbb{R}$. Denoting the Legendre transform of f by

$$I(s) \equiv \sup_{\theta \in \mathbb{R}} [\theta s - f(\theta)], \qquad s \in [-1, 1] \tag{2.9}$$

one can show (see, for example, ref. 23) that the following large-deviation estimate holds:

$$\lim_{\Lambda \nearrow \mathbb{Z}^{d+1}} \frac{1}{|\Lambda|} \log\{\mathrm{Prob}[M_\Lambda(\underline{\sigma}) \in (a, b)]\} = -\inf_{a<s<b} I(s) \tag{2.10}$$

We note here that the measure which describes the state with the magnetization $M \neq m$ is the Gibbs measure μ_v^b, obtained by adding a constant magnetic field b to the ESM, that is, changing H_x into $H_x + b\sigma_x$. This measure will in general not describe the space-time paths of a PCA of the type considered here, that is, $P_b(d\underline{\sigma}_n \mid \underline{\sigma}_{n-1})$ will not be a product measure. We do not know at the present time, however, whether μ_v^b corresponds to some more general Markov process on the state space $\{-1, 1\}^{\mathbb{Z}^d}$, that is, whether μ_v^b has the global Markov property in the time direction. An easy exception to the above general rule is the case in which only two sets, corresponding to the empty set and to a single site, occur in (1.3): $A_1 = \varnothing$, $A_2 = \{i\}$. In this case, which always occurs when $d = 0$ and μ_v thus describes a one-dimensional Markov chain on the state space $\{-1, 1\}$, the Gibbs state μ_v^b will still describe a Markov chain.

Relations similar to (2.10) hold also for other observables. It is not clear, however, to us at present how to extend (2.10) to incorporate statements about *equal time* or *equal site* events for the PCA, since that would require special choices for the volumes Λ. There is, however, a certain regime for the PCA in which we have a more complete picture, and in which, for example, the results of refs. 25 and 26 about the asymptotics of occurrence times of rare events for PCAs can be recovered by applying the large-deviation theory for Gibbs states. We discuss this briefly at the end of Section 4.

3. INEQUALITIES

When all the r_A's in (1.4) are nonnegative, which *ipso facto* implies that $\sum r_A \leq 1$, we say that the PCA has *positive dynamics*. It is then easy to show[5] that if the measure ρ on spin configurations $\underline{\sigma}_{-N}$ has positive correlations, i.e., $\langle \sigma_B \rangle_\rho \geqslant 0$ for any finite $B \subset \mathbb{Z}^d_{-N}$, then the measure μ_ρ^{-N} on spin configurations in the future of $-N$ also has positive correlations:

$$\langle \sigma_C \rangle_{\mu_\rho^{-N}} \geqslant 0, \qquad C \subset \bigcup_{n \geqslant -N} \mathbb{Z}^d_n \tag{3.1}$$

Furthermore, the $\langle \sigma_C \rangle_{\mu_\rho^{-N}}$ are nondecreasing in the r_A's and in the initial correlations. In particular, if ρP^n converges, as $n \to \infty$, to a (stationary) measure ν, then

$$\langle \sigma_C \rangle_{\mu_\nu} \geqslant 0 \tag{3.2}$$

for all $C \subset \mathbb{Z}^{d+1}$.

It follows from (3.2) and the identity $\langle \sigma_0 \rangle_{\mu_\nu} = \langle h_0 \rangle_{\mu_\nu}$ that if

$$h_0(\underline{\eta}) = \tilde{r}_0 + \tilde{r}_1 \sum_{i \in U} \eta_i + \sum_{|A| \geqslant 2} r_A \eta_A \tag{3.3}$$

we have

$$\langle \sigma_0 \rangle_{\mu_\nu} \equiv m \geqslant \frac{\tilde{r}_0}{1 - |U| \, \tilde{r}_1} \equiv \lambda \tag{3.4}$$

As a further consequence, (1.33) implies

$$\hat{s}(\mu_\nu) \leqslant f((1+\lambda)/2) + f((1-\lambda)/2) \tag{3.5}$$

Positive dynamics for the PCA will also imply positivity of the correlations for the Gibbs state μ_Γ of the Hamiltonian (1.6) in a cube $\Gamma \subset \mathbb{Z}^{d+1}$ with the periodic boundary conditions in the space directions, $+1$ boundary conditions on the top, and free boundary conditions on the bottom. This follows from the simple observation that the expectation value of σ_C, $C \subset \Gamma$, in such a state can be written as

$$\langle \sigma_C \rangle_{\mu_\Gamma} = 2^{-|\Gamma|} \sum_{\sigma_y = \pm 1,\, y \in \Gamma} \left[\sigma_C \prod_{x \in \Gamma} \left(1 + \sigma_x \sum_A r_A \sigma_{A+x} \right) \right] \tag{3.6}$$

which is clearly nonnegative for $r_A \geqslant 0$. In (3.6) the translation has to be interpreted in a periodic way, and the σ_x's in the top layer set to $+1$. Clearly the same will hold for any other domain with boundary conditions leading to a structure like (3.6) and for any infinite-volume limits of such states. It also follows from (3.6) that

$$\langle \sigma_x \sigma_{B+x} \rangle_{\mu_\Gamma} \geqslant r_B \tag{3.7}$$

for any $B \subset \Gamma$. One may generalize (3.7) to obtain improved lower bounds on $\langle \sigma_C \rangle$ by considering any number of terms in the expansion of the product in (3.6).

The equation

$$\langle \sigma_C \rangle \geqslant 0 \tag{3.8}$$

is just the *first Griffiths inequality*[27] for ferromagnetic equilibrium systems, i.e., those for which $J_R \geqslant 0$, for all R. For a PCA with positive dynamics the part of $H_0(\sigma_0, \underline{\eta})$ corresponding to $\sigma_0 Q_0(\underline{\eta})$ will in fact be ferromagnetic, but the other part, $\log\cosh Q_0(\underline{\eta})$, will have negative J_R's, so that the PCA inequality is an independent result. The converse of the first statement above is not true: $h_0(\underline{\eta}) \equiv -\tanh Q_0(\underline{\eta})$ does not necessarily define a positive dynamics even when $Q_0(\underline{\eta}) = -\sum_R J_R \eta_R$ with $J_R \geqslant 0$.

A positive dynamics thus shares some properties with ferromagnets. In fact, replacing $r_B \sigma_{B+x}$ in (3.6) by $(r_B + \tilde{r}_{B+x})\,\sigma_{B+x}$, taking the derivative with respect to $\tilde{r}_{B+x}$, and then setting $\tilde{r}_{B+x}$ to zero, we obtain

$$\frac{\partial \langle \sigma_C \sigma_x \rangle}{\partial \tilde{r}_{B+x}} = \frac{1}{2} \langle \sigma_C \sigma_{B+x} e^{H_x(\sigma)} \rangle \geqslant 0 \tag{3.9}$$

Summing over all sets B yields

$$\sum_B \frac{\partial \langle \sigma_C \rangle}{\partial \tilde{r}_{B+x}} r_B = \left\langle \sigma_C \left(1 - \frac{1}{2} e^{H_x(\sigma)} \right) \right\rangle \geqslant 0 \tag{3.10}$$

These inequalities have the flavor of the second Griffiths inequality $(\partial \langle \sigma_C \rangle / \partial J_B = \langle \sigma_C \sigma_B \rangle - \langle \sigma_C \rangle \langle \sigma_B \rangle \geqslant 0)$, but so far we have not found any interesting applications of (3.9) and (3.10). Note that it was shown in ref. 5 that the condition $\langle \sigma_A \rangle_\rho \geqslant 0$ is not sufficient to ensure the validity of the second Griffiths inequality for PCA with positive dynamics, although the possibility of its general validity when ρ is the stationary measure was not ruled out. It follows from ref. 28, however, that if we replace H_0 in (1.15) by

$$H_0^\varepsilon(\sigma_0, \underline{\eta}) = \sigma_0 Q_0(\underline{\eta}) + \varepsilon \log[2\cosh Q_0(\underline{\eta})] \tag{3.11}$$

then the resulting equilibrium system (which will not be equivalent to a PCA for $\varepsilon \neq 1$) will satisfy both Griffiths inequalities whenever $\sigma_0 Q_0(\underline{\eta})$ is ferromagneic and ε is sufficiently small. We discuss these points further in Section 6.

3.1. FKG Inequalities

We say that a function $f(\underline{\sigma}_A)$ is *nondecreasing* if $f(\underline{\sigma}_A) \geqslant f(\underline{\eta}_A)$ whenever $\sigma_i \geqslant \eta_i$ for all $i \in A$; see ref. 13. A PCA is *attractive* if the function $h_0(\sigma)$

in (1.3) is nondecreasing. An attractive dynamics has the property that it preserves positive correlations between nondecreasing functions, i.e., if the initial measure ρ satisfies the FKG inequalities[29]

$$\langle fg\rangle_\rho \geq \langle f\rangle_\rho \langle g\rangle_\rho \tag{3.12}$$

for any two nondecreasing f and g, then so does the measure on space-time configurations, i.e.,

$$\langle FG\rangle_{\mu_\rho^{-N}} \geq \langle F\rangle_{\mu_\rho^{-N}} \langle G\rangle_{\mu_\rho^{-N}} \tag{3.13}$$

with F, G again nondecreasing. Following ref. 30, one can then derive bounds on the decay of the correlations between two widely separated sets (in space and time) of spins in terms of the decay of time-displaced pair correlations. For more details see ref. 5.

The proof of (3.13) is based on the observations that: (i) for any fixed $\underline{\sigma}_n$, $P(d\underline{\sigma}_n \mid \underline{\sigma}_{n-1})$ is a product measure which satisfies the FKG inequalities, and (ii) for an attractive dynamics, $\int P(d\underline{\sigma}_n \mid \underline{\sigma}_{n-1}) f(\underline{\sigma}_n, \underline{\sigma}_{n-1},\ldots, \underline{\sigma}_{n-k})$ is increasing whenever f is increasing. We note also that (3.12) is satisfied for $\rho = \delta_\eta$, the measure concentrated on the configuration η. A particular consequence of attractive dynamics is that, for such PCA, ergodicity is equivalent to

$$\lim_{n\uparrow\infty} [\langle \sigma_0\rangle_{\delta_+ P^n} - \langle \sigma_0\rangle_{\delta_- P^n}] = 0 \tag{3.14}$$

where δ_+ (respectively δ_-) is the delta measure on the configuration with all spin values equal to 1 (respectively -1).[13,31]

4. THE HIGH-NOISE REGIME

The *high-temperature* regime in an ESM with finite-range interactions is characterized by the uniqueness of the Gibbs state and by exponential decay of correlations and their analytic dependence on the interaction parameters.[32] The PCA–ESM correspondence implies that any such high-temperature theory for the Gibbs state will find its equivalent as a high-noise theory for PCAs [or more precisely, for those PCAs which satisfy (1.9) and hence for which the correspondence is well defined]. In this regime the well-established ESM theory tells us a great deal about the properties of the state μ_ν, including the high-temperature expansion of the $(d+1)$-dimensional correlation functions,[33] and implies that the state ν itself is Gibbsian, as we discuss below. Here, however, we will take another route to obtain some of these results for the PCA.

The ESM high-temperature regime corresponds to the smallness of all interaction coefficients J_R, with $|R| \geqslant 2$, in the Hamiltonian (1.15); a complete characterization is given by the criterion of Dobrushin and Shlosman.[34] For ESM which arise from a PCA, however, the J_R are not independent [due to the normalization condition (1.12)] and therefore other, simpler high-T criteria (for example, the Dobrushin single-site condition[35]) on the interactions may be unnecessarily restrictive in the PCA context. For this reason, and because the correspondence is not valid for all PCAs, we will formulate *high-noise* conditions for PCAs directly in terms of the r_A. The conditions we will give do not in general correspond to high temperature in the usual sense (some J_R may be infinite), but they will give rise to a behavior of the PCA which is relatively simple; in particular, which is close to a situation of independently evolving spins. We will return below to the high-temperature regime defined by the Dobrushin–Shlosman criterion when we discuss the Gibbsian nature of the PCA measure ν.

4.1. High-Noise Conditions

We give two independent conditions, either of which may serve to characterize a high-noise regime.

Condition 1:

$$\delta_1 \equiv \sum_A |r_A| < 1 \tag{4.1}$$

This kind of condition first appeared in the work of Holley and Stroock,[36] where, using duality, (4.1) was shown to give exponential ergodicity of the corresponding continuous-time process. A discussion of this approach is contained in Liggett's book.[13] Condition (4.1) was also discussed in the context of PCAs in ref. 5. The proof we present below uses (4.1) in a *direct* way, i.e., without using any duality or domination arguments.

Note that the condition (4.1) is automatically satisfied for any *three-input* dynamics of the form

$$h_0(\underline{\sigma}) = \tilde{r}_1 \sigma_{A_1} + \tilde{r}_2 \sigma_{A_2} + \tilde{r}_3 \sigma_{A_3} \tag{4.2}$$

with $\|h_0\| < 1$. By (1.22)–(1.23) the same conclusion holds when Q_0 in (1.15) involves only two terms; absence of spontaneous magnetization when these two terms are single spins and the system is symmetric under spin inversion was obtained independently in ref. 18. Similarly, any PCA with positive dynamics, as defined in Section 3, verifies $\delta_1 < 1$, whenever the configuration $\sigma_{n,i} \equiv 1$ is not a trap.

Condition 2:

$$\delta_2 \equiv \sum_j \sup_{\underline{\sigma}} \left| \sum_{A \ni j} r_A \sigma_A \right| < 1 \tag{4.3}$$

This is a discrete version of the well-known $M < \varepsilon$ or Dobrushin condition for exponential ergodicity of interacting particle systems.[13,37,38] Note that

$$\begin{aligned}\delta_2 &= \sum_j \sup_{\underline{\sigma}} |h_0(\underline{\sigma}^j) - h_0(\underline{\sigma})|/2 \\ &= \sum_j \sup_{\underline{\sigma}} |p_0(\cdot \mid \underline{\sigma}^j) - p_0(\cdot \mid \underline{\sigma})| \end{aligned}\tag{4.4}$$

where $\underline{\sigma}^j$ is the configuration obtained from $\underline{\sigma}$ by flipping the spin at site j. It is easy to see from (4.4) that the Dobrushin single-site condition[35] for the Gibbs measure μ_ν is always more restrictive than condition (4.3): in (4.4), only those variations in the probability distribution of σ_0 are considered which are due to changing a spin value in the *past*. Moreover, (4.3) may be verified for dynamics possessing a trapping configuration (for example, in the BG model discussed in Section 5 with $p = 0$) and for which, as a consequence, (1.9) is violated and the PCA–ESM correspondence is singular. Steif[39] has independently obtained a stronger convergence result under the assumption (4.3), from which he shows the space-time Bernoullicity of the PCA (in the sense of ergodic theory, with respect to translations in $\mathbb{Z}^{d+1}$).

4.2. Results

We will show that condition (4.1) or (4.3) produces an exponential convergence to a unique stationary state. It then follows quite easily that this state has exponential decay of correlations. To get a more formal statement of these facts, we need to introduce some notation. We will, however, postpone defining the appropriate seminorms, etc., until we actually give the proofs of the various theorems.

In this section $\underline{\sigma}$ will denote a spin configuration on $\mathbb{Z}^d$, not $\mathbb{Z}^{d+1}$. Let $f(\underline{\sigma})$ denote a function of the spin configuration in a finite region, say $A \subset \mathbb{Z}^d$. The PCA defines, via (1.1), the operation

$$Pf(\underline{\eta}) \equiv \int f(\underline{\sigma})\, P(d\sigma \mid \underline{\eta}) \tag{4.5}$$

Note that $P^n f(\underline{\eta})$ is the expectation value of the function f at time n if the PCA was started at time zero from configuration $\underline{\eta}$. We write

$$\|F\| = \sup_{\underline{\eta}} |F(\underline{\eta})| \tag{4.6}$$

Theorem 1. Suppose that either (4.1) or (4.3) holds. Then the PCA is ergodic and

$$\|P^n f - \langle f \rangle_\nu\| \leqslant \frac{2c(f)}{1-\delta}\,\delta^n \tag{4.7}$$

where ν is the unique invariant measure, $\delta = \delta_1$ or $\delta = \delta_2$ according to whether (4.1) or (4.3), respectively, is satisfied, and $c(f) < \infty$ is a constant which depends on the function f.

Since we have restricted ourselves to finite-range PCA, there is some integer $r < \infty$ such that $h_0(\underline{\sigma})$ does not depend on any σ_i for which $|i| \geqslant r$, where $|i| = \sum_{k=1}^{d} |i_k|$ for $i = (i_1,..., i_d) \in \mathbb{Z}^d$. Let now $g(\underline{\sigma})$ be another function, depending on the configuration of spins in the finite region $B \subset \mathbb{Z}^d$. We write $d(A, B)$ for the distance between the two sets A and B.

Theorem 2. Under the same hypothesis as for Theorem 1 and using the same notation,

$$\|P^n fg - P^n f P^n g\| \leqslant \frac{6c(f)\,c(g)}{1-\delta}\,\delta^{d(A,B)/2r} \tag{4.8}$$

and, in particular, $|\langle fg \rangle_\nu - \langle f \rangle_\nu \langle g \rangle_\nu|$ satisfies the same bound (4.8).

Proofs. Let A be any finite region in $\mathbb{Z}^d$ and define, for every continuous [in the uniform topology given by the sup norm (4.6)] function F, the coefficient $\lambda_A(F)$ given by the expectation of $F(\underline{\sigma})\,\sigma_A$ in the product measure which assigns equal weights to $\sigma = \pm 1$. Thus

$$\lambda_A(F) \equiv \prod_i \left[\frac{1}{2} \sum_{\sigma_i = \pm 1}\right] F(\underline{\sigma})\,\sigma_A \tag{4.9}$$

and formally $F = \sum_A \lambda_A(F)\,\sigma_A$; for example, $\lambda_A(h_0) = r_A$. We denote by D_1 the space of continuous functions F for which

$$c_1(F) = \sum_{A \neq \varnothing} |\lambda_A(F)| < \infty \tag{4.10}$$

Following Dobrushin and others,[13,35,38] we define the *variation* of F under a spin flip at site $i \in \mathbb{Z}^d$ by

$$\Delta_F(i) \equiv \sup_{\underline{\sigma}} |F(\underline{\sigma}^i) - F(\underline{\sigma})| \tag{4.11}$$

and the corresponding space D_2 of continuous functions F for which

$$c_2(F) \equiv \sum_{i \in \mathbb{Z}^d} \Delta_F(i) < \infty \tag{4.12}$$

Lemma 1. If $\delta_1 \leqslant 1$, then, for all $f \in D_1$,

$$c_1(Pf) \leqslant \delta_1 c_1(f) \tag{4.13}$$

Proof. Since $f(\underline{\sigma}) = \sum_A \lambda_A(f)\, \sigma_A$, we have

$$Pf(\underline{\eta}) = \sum_A \lambda_A(f) \prod_{i \in A} h_i(\underline{\eta}) \tag{4.14}$$

Therefore,

$$\lambda_B(Pf) = \sum_A \lambda_A(f)\, M_{AB} \tag{4.15}$$

with

$$M_{AB} = \lambda_B\left(\prod_{i \in A} h_i\right) \tag{4.16}$$

for any finite A, $B \subset \mathbb{Z}^d$. If A is the empty set, then (4.16) is zero unless B is also empty:

$$M_{\varnothing B} = \delta_{\varnothing B} \tag{4.17}$$

If $A = \{i_1, ..., i_m\}$ is not empty, then some computation shows that

$$M_{AB} = \sum{}' \prod_{l=1}^{m} r_{K_l} \tag{4.18}$$

where the primed sum is over all sets $K_1, ..., K_m$ such that

$$(K_1 + i_1)\, \Delta\, (K_2 + i_2)\, \Delta \cdots \Delta\, (K_m + i_m) = B \tag{4.19}$$

for Δ the symmetric difference of sets. Hence,

$$\sum_{B \neq \varnothing} |M_{AB}| \leqslant \sum_{K_1, ..., K_m} \prod_{l=1}^{m} |r_{K_l}| = \delta_1^m \tag{4.20}$$

Combining (4.17) and (4.20) with (4.15) gives the desired bound (4.13). ∎

Lemma 2. For all $f \in D_2$,

$$c_2(Pf) \leqslant \delta_2 c_2(f) \tag{4.21}$$

Proof. For $i \in \mathbb{Z}^d$,

$$Pf(\underline{\eta}') - Pf(\underline{\eta}) = \int f(\underline{\sigma})[P(d\underline{\sigma} \mid \underline{\eta}') - P(d\underline{\sigma} \mid \underline{\eta})] \tag{4.22}$$

Clearly,

$$P(d\underline{\sigma}\,|\,\underline{\eta}^i) - P(d\underline{\sigma}\,|\,\underline{\eta})$$
$$= P(d\underline{\sigma}_{R(i)^c}\,|\,\underline{\eta})\left[\prod_{j\in R(i)} p_j(\sigma_j\,|\,\underline{\eta}^i) - \prod_{j\in R(i)} p_j(\sigma_j\,|\,\underline{\eta})\right]\rho_0(d\underline{\sigma}_{R(i)}) \tag{4.23}$$

where $R(i) \equiv \{j\in\mathbb{Z}^d\,|\,p_j(\cdot\,|\,\underline{\eta}^i) - p_j(\cdot\,|\,\underline{\eta}) \neq 0\}$ and ρ_0 is a counting measure. The bracketed factor in (4.23) can be written as a telescoping sum by introducing an arbitrary order in $R(i)$:

$$\prod_{j\in R(i)} p_j(\sigma_j\,|\,\underline{\eta}^i) - \prod_{j\in R(i)} p_j(\sigma_j\,|\,\underline{\eta})$$
$$= \sum_{j\in R(i)}\left[\prod_{\substack{k\in R(i)\\ k<j}} p_k(\sigma_k\,|\,\underline{\eta}^i)\right][p_j(\sigma_j\,|\,\underline{\eta}^i) - p_j(\sigma_j\,|\,\underline{\eta})]$$
$$\times\left[\prod_{\substack{k\in R(i)\\ k>j}} p_k(\sigma_k\,|\,\underline{\eta})\right] \tag{4.24}$$

Substituting (4.24) in (4.23) and using (4.22) gives

$$\Delta_{Pf}(i) \leqslant \frac{1}{2}\sum_{j\in R(i)} \sup_{\underline{\eta}} |h_j(\underline{\eta}^i) - h_j(\underline{\eta})|\,\Delta_f(j) \tag{4.25}$$

The bound (4.21) is now easily obtained from (4.25) by summing over $i\in\mathbb{Z}^d$. ∎

Given Lemmas 1 and 2, the proofs of Theorems 1 and 2 are standard; see, for example, ref. 13. We give them here for completeness. From now on we use the convention that we write δ, D, and $c(f)$ instead of δ_1, D_1, and $c_1(f)$ or instead of δ_2, D_2, and $c_2(f)$ according to which of the conditions (4.1) or (4.3) is satisfied.

Proof of Theorem 1. From

$$\|(P-1)f\| = \|(P-1)[f - \lambda_{\varnothing}(f)]\| \leqslant 2c(f)$$

and Lemmas 1 and 2, it follows that, for any $m > n > 0$,

$$\|P^m f - P^n f\| = \left\|\sum_{k=n}^{m-1} (P-1)\,P^k f\right\| \leqslant 2\sum_{k=n}^{m-1} c(P^k f) \leqslant \frac{2c(f)}{1-\delta}\,\delta^n \tag{4.26}$$

Hence, $\lim_{n\to\infty} P^n f$ exists. The bound (4.13) or (4.21) implies then that $\lim_n P^n f$ must be constant. On the other hand, for all $n = 0, 1, 2,\ldots,$

$$\langle P^n f\rangle_\nu = \langle f\rangle_\nu \tag{4.27}$$

for any invariant measure ν. It follows that

$$\lim_n P^n f = \langle f \rangle_\nu \tag{4.28}$$

for all $f \in D$. Equation (4.27) determines ν uniquely, since the space D is dense in the space C of continuous functions. Equation (4.6) follows by taking the $m \to \infty$ limit in (4.26) and using (4.28). ∎

Proof of Theorem 2. Let N be the smallest integer greater than or equal to $d(A, B)/2r$. Then, for all $n \leqslant N$,

$$P^n fg - P^n f P^n g = 0 \tag{4.29}$$

Take now $n > N$. Then from (4.26),

$$\begin{aligned} &\|P^n fg - P^n f P^n g\| \\ &\quad \leqslant \|P^N fg - P^N f P^N g\| + \|P^n fg - P^N fg\| \\ &\qquad + \|P^n f\| \cdot \|P^n g - P^N g\| + \|P^N g\| \cdot \|P^n f - P^N f\| \\ &\quad \leqslant \frac{2c(fg)}{1-\delta}\,\delta^N + \frac{2c(g)\,\|f\|}{1-\delta}\,\delta^N + \frac{2c(f)\,\|g\|}{1-\delta}\,\delta^N \end{aligned} \tag{4.30}$$

It has no effect on the lhs or the rhs of (4.30) to assume that $\lambda_\varnothing(f) = \lambda_\varnothing(g) = 0$. For such functions, $\|fg\| \leqslant c(f)\, c(g)$, $\|f\| \leqslant c(f)$, and $\|g\| \leqslant c(g)$. Substituting these inequalities in (4.30) and using $N \geqslant d(A, B)/2r$ yields (4.8). The rest of Theorem 2 is an immediate consequence of Theorem 1. ∎

We remark that the assumptions of translation invariance and of finite range are not essential for the arguments presented here. The appropriate conditions in a more general context are easy to reconstruct; see also Section 6.

4.3. Gibbsian Nature of the PCA Measure at High Temperature

We now turn to the question of whether the measure ν itself is a Gibbs measure for some Hamiltonian on spin configurations $\underline{\eta}$. This question is in general very difficult; we refer to Künsch[40] for a discussion of the continuous-time case. For PCAs which satisfy (1.9), so that the PCA–ESM correspondence is well defined, the question can be answered in the affirmative in the Dobrushin–Shlosman high-temperature regime for the ESM. We begin by recalling the definition of this regime.

We write $\underline{\zeta}$ for a spin configuration on $\mathbb{Z}^{d+1}$ and $\underline{\zeta}_V$ for the restriction of this configuration to $V \subset \mathbb{Z}^{d+1}$. Let $W \subset V$ be finite subsets of $\mathbb{Z}^{d+1}$, and for any $\underline{\zeta}$ let $\mu_{V,W}(\cdot \mid \underline{\zeta}_{V^c})$ be the ESM Gibbs measure in W induced by imposing $\underline{\zeta}_{V^c}$ as boundary conditions outside V, that is, for $X \subset \{+1, -1\}^W$,

$$\mu_{V,W}(X \mid \underline{\zeta}_{V^c}) = Z^{-1} \sum_{\{\underline{\zeta}_V \mid \underline{\zeta}_W \in X\}} \exp\left[-\sum_{x \in V} H_x(\zeta_x, \underline{\zeta}) \right] \tag{4.31}$$

Then the ESM is in the Dobrushin–Shlosman regime if and only if there exist constants K, $\gamma > 0$ such that, for any V and W as above and any $\underline{\zeta}^1$ and $\underline{\zeta}^2$ for which $\underline{\zeta}^1_{V^c}$ and $\underline{\zeta}^2_{V^c}$ differ only at some site $x \in V^c$, we have

$$|\mu_{V,W}(X \mid \underline{\zeta}^1_{V^c}) - \mu_{V,W}(X \mid \underline{\zeta}^2_{V^c})| \leqslant K e^{-\gamma \operatorname{dist}(W,x)} \tag{4.32}$$

Every ESM satisfies (4.32) at sufficiently high temperature and (4.32) implies that the Gibbs state in infinite volume is unique. Finally, it follows immediately from (4.32) that there exist K' and γ' such that, when $\underline{\zeta}^1_{V^c}$ and $\underline{\zeta}^2_{V^c}$ differ on $U \subset V^c$,

$$|\mu_{V,W}(X \mid \underline{\zeta}^1_{V^c}) - \mu_{V,W}(X \mid \underline{\zeta}^2_{V^c})| \leqslant K' \, |W| \, e^{-\gamma' \operatorname{dist}(W,U)} \tag{4.33}$$

Now we can state the following result.

Theorem 3. Suppose that the PCA satisfies (1.9), and that the corresponding ESM is in the Dobrushin–Shlosman regime. Then the stationary measure ν is unique and is a Gibbs state.

Proof. We will regard ν as a measure on $\mathbb{Z}_0^d \subset \mathbb{Z}^{d+1}$; ν is unique because it is the marginal density of the unique Gibbs measure $\mu_\nu \equiv \mu$ for the ESM. Let $\underline{\xi}$ be an arbitrary spin configuration on $\mathbb{Z}_0^d$. To prove that ν is Gibbsian, it suffices, by ref. 41, to show that the conditional probabilities

$$\nu_0(\underline{\xi}) \equiv \operatorname{Prob}[\sigma_0 = \xi_0 \mid \sigma_j = \xi_j \text{ for all } j \in \mathbb{Z}_0^d \setminus \{0\}] \tag{4.34}$$

are strictly positive and continuous in $\underline{\xi}$. [The probability in (4.34) may be equivalently regarded as defined using either ν or μ.] The strict positivity is an immediate consequence of the assumption $\|h_0\| < 1$. To prove the continuity, we introduce a sequence $\{\Lambda_k\}$ of d-cubes in $\mathbb{Z}_0^d$ centered around the origin, with side length $2k$. The conditional probability

$$\nu_0^{(k)}(\underline{\xi}) = \operatorname{Prob}[\sigma_0 = \xi_0 \mid \sigma_j = \xi_j \text{ for } j \in \Lambda_k \setminus \{0\}] \tag{4.35}$$

is clearly continuous in $\underline{\xi}$; we will prove the uniform convergence of $\nu_0^{(k)}$ to ν_0, so that ν_0 is also continuous. Thus, we must show that for $k > l$ the differences

$$|\nu_0^{(k)}(\underline{\xi}) - \nu_0^{(l)}(\underline{\xi})| \tag{4.36}$$

are arbitrarily small for sufficiently large l, uniformly in $\underline{\xi}$ and k. This is equivalent to obtaining a uniform estimate for

$$|\nu_0^{(k)}(\underline{\xi}^1) - \nu_0^{(k)}(\underline{\xi}^2)| \tag{4.37}$$

with $\underline{\xi}^1 = \underline{\xi}^2 = \underline{\xi}$ on Λ_l.

Now introduce a large $(d+1)$-cube $V_m \subset \mathbb{Z}^{d+1}$ centered at the origin, with side length $2m$ satisfying $m > k$. For $X \subset \{+1, -1\}^{\Lambda_k}$,

$$\mu(X) = \nu(X) = \int \mu_{V_m, \Lambda_k}(X \mid \underline{\zeta}_{V_m^c})\, \mu(d\underline{\zeta}) \tag{4.38}$$

By (4.33), however, $\mu_{V_m,\Lambda_k}(\cdot \mid \underline{\zeta}_{V_m^c})$ depends only weakly on $\underline{\zeta}$ for large m, so that, by (4.38), we may calculate probabilities or conditional probabilities in Λ_k to order $e^{-\gamma'(m-k)}$ using $\mu_{V_m,\Lambda_k}(\cdot \mid \underline{\zeta}_{V_m^c})$, for any fixed $\underline{\zeta}$. Thus, if $W_m = (V_m \backslash \Lambda_k) \cup \{0\}$ and $\underline{\zeta}^1$ and $\underline{\zeta}^2$ agree with $\underline{\xi}^1$ and $\underline{\xi}^2$, respectively, in Λ_k and with each other in Λ_k^c, the above remark implies that

$$|\nu_0^{(k)}(\underline{\xi}^i) - \mu_{W_m,\{0\}}(\{\zeta_0 = \xi_0\} \mid \underline{\zeta}^i_{W_m^c})| \leqslant K_1 e^{-\gamma'(m-k)} \tag{4.39}$$

Another application of (4.33) yields

$$|\mu_{W_m,\{0\}}(\{\zeta_0 = \xi_0\} \mid \underline{\zeta}^1_{W_m^c}) - \mu_{W_m,\{0\}}(\{\zeta_0 = \xi_0\} \mid \underline{\zeta}^2_{W_m^c})| \leqslant K' e^{-\gamma' l} \tag{4.40}$$

Estimates (4.39) and (4.40) lead to the desired uniform bound on (4.37). ∎

The Gibbsian nature of the invariant measure ν implies, among other things, that a more complete fluctuation theory is available for ν than would apply in general. In particular, one may use large-deviation theory for Gibbs states to obtain results similar to those of refs. 25 and 26. This is in contrast to the general situation discussed in Section 2, where we could consider only large deviations in the space-time measure μ_ν.

5. PHASE TRANSITIONS AND THE TOOM MODEL

Our goal in this section is to describe several PCAs for which at least two phases exist at low noise levels. The particular models that we study appear to be most conveniently defined on a more general class of lattice geometries than we considered earlier. To motivate this development, we consider first a well-known example: implemention of parallel Monte Carlo dynamics for an equilibrium lattice system by alternate updating of sites on two sublattices.

Example 5.1. Let $\mathbb{L}^{(0)}$ be a d-dimensional lattice which is the union of two disjoint sublattices $\mathbb{L}^{(+)}$ and $\mathbb{L}^{(-)}$, arranged so that nearest neighbor sites lie in different sublattices; for example, take $\mathbb{L}^{(0)} = \mathbb{Z}^d$. Suppose that $\mathscr{H}^{(0)}$ is a spin Hamiltonian on $\mathbb{L}^{(0)}$ containing only one-body terms or two-body nearest-neighbor interactions:

$$\mathscr{H}^{(0)} = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j - b \sum_i \sigma_i \tag{5.1}$$

We define a stochastic evolution of the spin configuration on $\mathbb{L}^{(0)}$ in which spins on the sublattices are updated at alternate times, with probabilities given by the Gibbs measure at inverse temperature β conditioned on their neighbors. Thus, if n is even, the updating rule is

$$p(\sigma_{n,i} \mid \underline{\sigma}_{n-1}) = \begin{cases} \dfrac{\exp[-\beta \sigma_{n,i} Q_i(\underline{\sigma}_{n-1})]}{2 \cosh \beta Q_i(\underline{\sigma}_{n-1})}, & \text{if} \quad i \in \mathbb{L}^{(+)} \\ \delta_{\sigma_{n,i}\sigma_{n-1,i}}, & \text{if} \quad i \in \mathbb{L}^{(-)} \end{cases} \tag{5.2}$$

where

$$Q_{n,i}(\underline{\eta}) = -J \sum_{j \in U(i)} \eta_j - b \tag{5.3}$$

for $U(i)$ the set of nearest neighbors of i. If n is odd, the updating rule is obtained by reversing the roles of $\mathbb{L}^{(+)}$ and $\mathbb{L}^{(-)}$. Then an easy calculation verifies that any Gibbs measure $\nu^{(0)}$ for the d-dimensional system with Hamiltonian (5.1) is invariant under this evolution.

One could view the space-time histories of this process as spin configurations on $\mathbb{Z} \times \mathbb{L}^{(0)}$, but it is also natural to ignore the sites at which the spin is unchanged, and consider these histories to be defined on the lattice

$$\mathbb{L} = [(2\mathbb{Z}) \times \mathbb{L}^{(+)}] \cup [(2\mathbb{Z}+1) \times \mathbb{L}^{(-)}] \subset \mathbb{R}^{d+1} \tag{5.4}$$

where $2\mathbb{Z}$ and $2\mathbb{Z}+1$ are the even and odd integers, respectively. For example, if $\mathbb{L}^{(0)}$ is $\mathbb{Z}$ with $\mathbb{L}^{\pm}$ the odd and even integers, then $\mathbb{L}$ is the diamond lattice shown in Fig. 1. Similar examples are discussed in ref. 7; see also

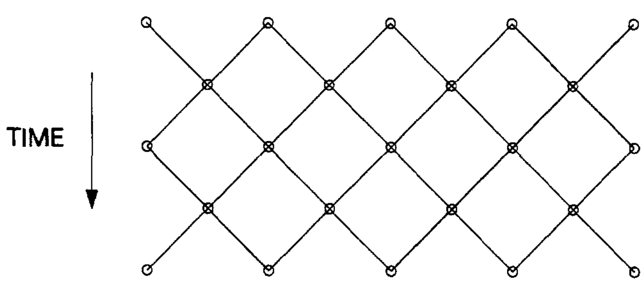

Fig. 1. Space-time lattice with $d = 1$ for PCA defined by alternate updating of odd and even sites in $\mathbb{Z}$.

Example 5.2 below for a discussion of a particular instance of this construction with $d=2$. Note that in the case $\mathbb{L}^{(0)}=\mathbb{Z}$ of Fig. 1 we may by a rotation identify $\mathbb{L}$ with $\mathbb{Z}^2$, but then the model is distinguished from the ESM on $\mathbb{Z}^2$ considered earlier by the fact that the time associated with a site $x=(i,j)\in\mathbb{Z}^2$ is $t(x)=i+j$.

The PCA described above can be viewed as an ESM on $\mathbb{L}$ in a natural way (a formal definition is given below). If $\nu^{(0)}$ is a Gibbs measure on $\mathbb{L}^{(0)}$ for $\mathscr{H}^{(0)}$, then the marginal measures $\nu^{(\pm)}$ on $\mathbb{L}^{(\pm)}$ are invariant under two time steps of the PCA. Moreover, $\nu^{(0)}$ itself may be identified with a measure on spin configurations on two consecutive layers $\mathbb{L}_n$, and gives rise to a Gibbs measure $\mu_{\nu^{(0)}}$ for the $(d+1)$-dimensional system on $\mathbb{L}$.

We now introduce a general geometry for the ESM–PCA correspondence which includes the construction above. Let $\mathbb{L}$ be a nondegenerate lattice in $\mathbb{R}^{d+1}$, that is, a discrete subset invariant under some $(d+1)$-dimensional group of translations. With each site $x\in\mathbb{L}$ we associate a time $t(x)$ given by $t(x)=x\cdot e$, where $e\in\mathbb{R}^{d+1}$ is some fixed vector, and we suppose that $t(x)$ assumes precisely all integer values:

$$\mathbb{L}=\bigcup_{n\in\mathbb{Z}}\{x\in\mathbb{L}\mid t(x)=n\}\equiv\bigcup_{n\in\mathbb{Z}}\mathbb{L}_n \tag{5.5}$$

Finally, we suppose that $\mathbb{L}$ is invariant under translation by $n_0 e$ for some (minimum, positive) integer n_0. Thus, $\mathbb{L}$ is formed by the periodic stacking of n_0 space lattices $\mathbb{L}_0, \mathbb{L}_1,\ldots, \mathbb{L}_{n_0-1}$.

Write $\underline{\sigma}\equiv(\sigma_x)_{x\in\mathbb{L}}$ for a spin configuration on $\mathbb{L}$ and $\underline{\sigma}_n$ for its restriction to $\mathbb{L}_n$. In analogy with (1.3), a PCA on $\mathbb{L}$ is specified by transition probabilities

$$p_x(\sigma_x\mid\underline{\sigma}_{t(x)-1})=\tfrac{1}{2}[1+\sigma_x h_x(\underline{\sigma}_{t(x)-1})] \tag{5.6}$$

which we assume to be of finite range and translation invariant; in typical examples, the p_x are actually invaiant under some transitive symmetry group of the lattice, so that the model is determined by p_0. The transition to a Gibbs system on $\mathbb{L}$ is as in (1.10)–(1.11):

$$\mathscr{H}(\underline{\sigma})=\sum_{x\in\mathbb{L}}H_x(\sigma_x,\underline{\sigma}_{t(x)-1}) \tag{5.7}$$

with

$$\exp[-H_x(\sigma_x,\underline{\sigma}_{t(x)-1})]\equiv p_x(\sigma_x\mid\underline{\sigma}_{t(x)-1}) \tag{5.8}$$

We will be interested in measures ν on $\{\underline{\sigma}_n\}$ which are invariant under n_0 steps of the PCA.

Example 5.2. *The Domany model.* Domany[1] has studied the construction of Example 5.1 with $\mathbb{L}^{(0)}$ the honeycomb lattice in $\mathbb{R}^2$ and $\mathbb{L}^{(\pm)}$ the two triangular sublattices, and with zero magnetic field. The lattice $\mathbb{L}$ is shown in Fig. 2; it is formed by stacking triangular plane lattices and, with rescaling of time, is isomorphic to the hexagonal close-packed (HCP) lattice in $\mathbb{R}^3$. The three corners of each shaded triangle in Fig. 2 determine, up to noise, the spin at the apex of the tetrahedron whose base is that triangle. Equation (5.2) leads to

$$h_0(\eta) = \tanh\left(\beta \sum_{j \in U} \eta_j\right) \tag{5.9}$$

with U the set of three sites in $\mathbb{L}_{-1}$ closest to 0 (we take $J = 1$ for simplicity); all other h_x are determined by translation invariance combined with invariance under the operation of translation of one unit in the time direction combined with a space reflection. The resulting ESM is itself an

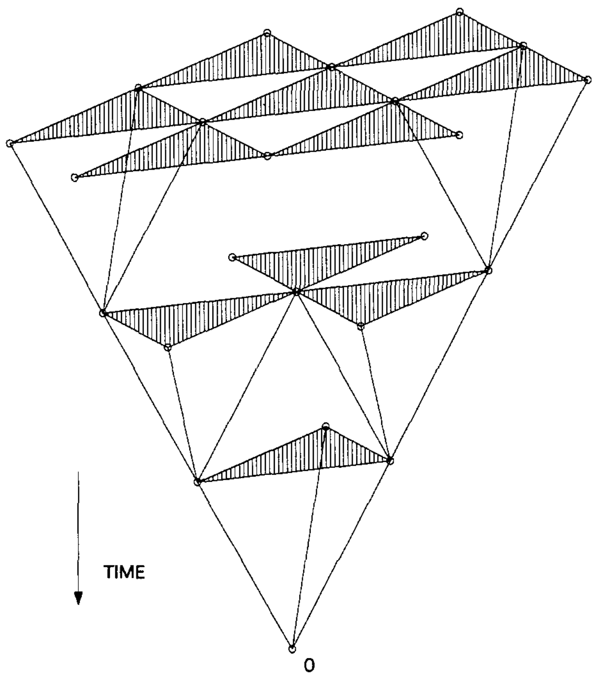

Fig. 2. HCP lattice for PCA defined by alternate updating of sites on triangular sublattices of hexagonal lattice. Sites at the vertices of each shaded triangle influence the site at the apex of the corresponding tetrahedron; all sites at time $t \geqslant -3$ which influence the origin are shown.

Ising model, with ferromagnetic couplings between lattice planes and antiferromagnetic nearest-neighbor couplings within the planes. Since the two-dimensional model is known to have a phase transition at some critical value of β, so does the ESM and hence also the PCA. We remark that it is easy to verify that the measures $\nu^{(\pm)}$ are Gibbs measures for a nearest-neighbor ferromagnetic Ising model on the triangular lattice.

5.1. The Toom model

We construct the Toom model on a lattice $\mathbb{L} = \mathbb{Z}^3$, but with time $t(x)$ associated with site x defined by $t(x) = -(x_1 + x_2 + x_3)$; that is, we take $e = -(1, 1, 1)$ above. The space lattices $\mathbb{L}_n$ are then triangular lattices, as in Example 5.2, but are arranged as follows: under projection perpendicular to the lattice planes, the sites of $\mathbb{L}_{n+1}$ coincide with the centers of alternate triangles in $\mathbb{L}_n$, i.e., of those with some fixed orientation; the sites of $\mathbb{L}_{n+2}$ coincide with the centers of the similarly oriented triangles in $\mathbb{L}_{n+1}$ and also with the centers of the oppositely oriented triangles in $\mathbb{L}_n$; finally, $\mathbb{L}_{n+3}$ coincides with $\mathbb{L}_n$ itself (so that $n_0 = 3$). If the time axis is rescaled, $\mathbb{L}$ becomes the face-centered-cubic (FCC) lattice. $\mathbb{L}$ is shown in Fig. 3.

We first define a *deterministic* Toom model. Transition probabilities are given by (5.6), with $h_x(\underline{\eta}) \equiv \phi_x(\underline{\eta})$, where

$$\phi_0(\underline{\eta}) = \operatorname{sgn}\left(\sum_{i \in U_0} \eta\right) \tag{5.10}$$

and $U_0 \equiv \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ consists of the three corners of the triangle in $\mathbb{L}_{-1}$ which lies directly above the origin. That is, by translation invariance, the spin at site x is determined by a majority vote of the spins at the corners of the triangle directly above. A *stochastic* Toom model is any (small) perturbation of this model, that is, a model on $\mathbb{L}$ of the general class defined by (5.6) and satisfying

$$\|h_0(\underline{\eta}) - \phi_0(\underline{\eta})\| < 2\varepsilon \tag{5.11}$$

The evolution in such a stochastic model follows the deterministic rule at each site with probability (at least) $1 - \varepsilon$ and does something else with probability (at most) ε. (Toom's treatment in fact allows for even more general perturbations.)

The deterministic dynamics clearly has $\sigma(x) \equiv +1$ and $\sigma(x) \equiv -1$ as invariant states. Moreover, these states are stable against finite excitations of spins of the opposite sign. In fact, such excitations shrink at a constant rate and hence disappear in finite time: the model is said to satisfy an

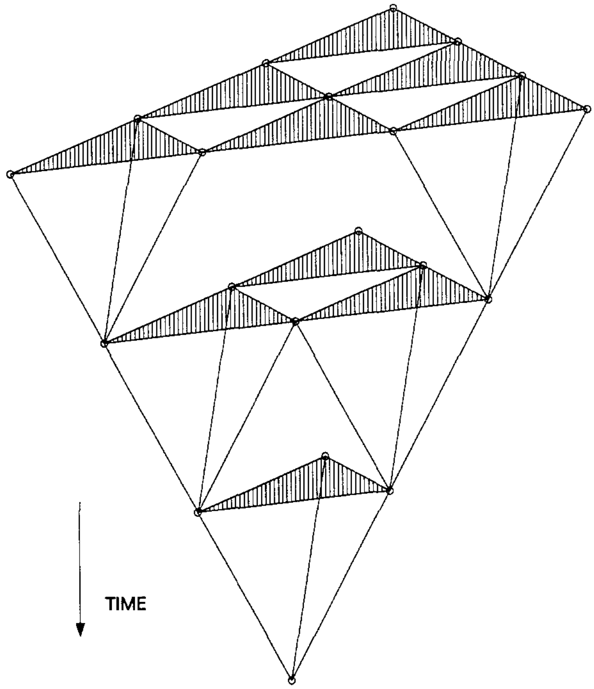

Fig. 3. FCC lattice for Toom model in symmetric presentation. Sites at the vertices of each shaded triangle influence the site at the apex of the corresponding tetrahedron; all sites at time $t \geqslant -3$ which influence the origin are shown.

eroder condition. This property is most easily seen for triangular excitations which at time zero have the form

$$\sigma_{ijk} = -1 \qquad \text{iff} \quad i, j, k \geqslant -N, \quad i + j + k = 0 \tag{5.12}$$

These excitations will have vanished by time $t = N$. Toom[9] has used this stability to show that the introduction of a small degree of noise does not remove the symmetry breaking, that is, if ε is sufficiently small, the process still has (at least) two invariant states, in one of which most spins are $+1$, in the other of which most are -1. We will review this proof in the Appendix.

We mention two specific possible choices of h_0:

(i) Bennett and Grinstein[19] study a particular choice of stochastic updating in which errors favoring up spins are made with a probability p, and errors favoring down spins with a probability q:

$$h_0(\underline{\eta}) = \begin{cases} (1-2p)\,\phi_0(\underline{\eta}) & \text{if} \quad \phi_0(\underline{\eta}) < 0 \\ (1-2q)\,\phi_0(\underline{\eta}) & \text{if} \quad \phi_0(\underline{\eta}) > 0 \end{cases} \tag{5.13}$$

The resulting ESM Hamiltonian $\mathscr{H}$ is the sum of single-site energies

$$H_0(\sigma_0, \underline{\sigma}_{-1}) = -(\beta\sigma_0 + J)\left(\sum_{j\in U_0} \sigma_{-1,j} - \prod_{j\in U_0} \sigma_{-1,j}\right) - b\sigma_0 - \gamma \tag{5.14}$$

where

$$\begin{aligned} \beta &= \frac{1}{8}\log\frac{(1-p)(1-q)}{pq}, & b &= \frac{1}{4}\log\frac{p(1-q)}{q(1-p)} \\ J &= \frac{1}{8}\log\frac{q(1-q)}{p(1-p)}, & \gamma &= \frac{1}{4}\log pq(1-p)(1-q) \end{aligned} \tag{5.15}$$

We will refer to this as the BG model, and we summarize the description of its phase plane below.

(ii) The geometry of one step of the Toom model is the same as that of one step of the Domany model of Example 5.2, that is, the updated spin at a site is determined (up to noise) by the spins at the corners of a triangle immediately above it. Moreover, the Domany updating rule (5.9) is in the Toom class. Using this updating rule on the Toom lattice therefore produces a PCA for which we know the invariant measures ν—they are just the measures $\nu^{(\pm)}$ discussed for the Domany model, that is, Gibbs measures for the 2D Ising model at zero magnetic field, on the triangular lattice. The existence of a phase transition in this model therefore follows either from Toom's theorem or from the known existence of a phase transition for the two-dimensional model. The Toom model defined with this updating is obtained from the model of Domany by the substitution of the FCC for the HCP lattice, that is, simply by a change in the way the triangular lattice planes are stacked, but the dynamics of the two models will be quite different. We note two particular features of this difference. First, at zero temperature the state $\sigma_x \equiv +1$ is stable on the FCC lattice against any finite excitation, but this is not true on the HCP lattice. For example, consider an excitation at time $t=-3$ for which three spins located at corners of an unshaded triangle in Figs. 2 and 3 have value -1. Then one each lattice a triangle of spins will have value -1 at $t=-2$. On the FCC lattice (Fig. 3) these will be located at corners of a shaded triangle and hence only the single spin directly below will have value -1 at $t=-1$; the excitation will disappear by time $t=0$. On the HCP lattice (Fig. 2), however, the spins at time $t=-2$ will again be at the corners of an unshaded triangle, so that the excitation will persist for all time. Second, we may modify the rule (5.9) to correspond to a nonzero magnetic field b on the honeycomb lattice:

$$h_0(\eta) = \tanh\beta\left(\sum_{j\in U}\eta_j + b\right) \tag{5.16}$$

On the FCC lattice the resulting PCA still has two invariant states, by Toom's theorem, but only one state exists for the Ising model on the honeycomb lattice and hence for the PCA on the HCP lattice.

We close this section with two remarks. First, the Toom model may be given an equivalent presentation on the conventional $\mathbb{Z}^3$ lattice (with an axis in the time direction) in which a site is updated according to the majority of spins among itself and its two neighbors to the north and east. Following Bennett and Grinstein, we will call this the *north-east-center* (NEC) presentation of the model, and refer to our original version as the *symmetric* presentation. Second, one may generalize the Toom model immediately to higher dimension; in space dimension d the deterministic rule for a site x is based on a vote of $(d+1)$ sites at the previous time. It is easy to modify the proof of Toom's theorem to show nonergodicity for small perturbations of this dynamics, even if the deterministic rule is strongly biased in favor of, say, $\sigma = +1$; it suffices to require that at least two $+1$ votes are needed to produce a $+1$ outcome.

5.2. The Phase Plane for the BG Model

The BG model is conveniently described in terms of the parameters $p+q$, the *noise*, and $(p-q)/(p+q)$, the *bias*. The phase plane is shown in Fig. 4 (taken in part from ref. 19). As Bennett and Grinstein point out, although noise and bias are somewhat analogous to temperature and magnetic field, two phases persist even in the presence of bias; in an Ising model, on the other hand, a nonzero magnetic field produces a unique

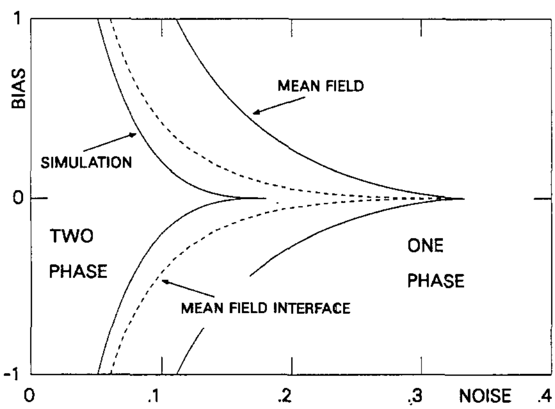

Fig. 4. Phase plane for BG model. Dashed lines are correction to mean-field phase transition line from consideration of interface motion.

phase. This may be understood from the ESM point of view as follows: the ESM Hamiltonian (5.14) contains one-, two-, three-, and four-spin interactions, and generic perturbations in the space of these interactions [e.g., the addition of a magnetic field term $b'\sigma_0$ to (5.14)] would indeed produce a unique phase. The special perturbations corresponding to PCA evolutions, however, including more general ones than the BG class, leave the system on the two-phase surface in the space of ESM Hamiltonians.

It appears, on the basis of the numerical simulations and some analytic evidence,[19,42] that the phase transition at zero bias ($p=q$) is a second-order transition, and that the transition at nonzero bias is first order. That is, when the phase transition line is approached at nonzero bias from within the two-phase region, the magnetization in the unfavored phase approaches a limit distinct from the magnetization in the favored phase; on the other hand, the favored phase appears to vary quite smoothly. Note in particular that when, say, the bias is 1 (i.e., $q=0$), one invariant state is known to have $\sigma_x \equiv +1$; this state certainly does not change as the noise is increased to shift the system across the phase transition line.

The free energy is, as mentioned earlier, identically zero for any Gibbs state of the ESM; in particular, for both pure phases in the two-state region and for the single phase in the one-phase region. Bennett and Grinstein suggest that immediately to the left of the (true) phase transition curve in Fig. 4 there exists a metastable phase with a prevalence of spins of the "wrong" sign, and with a free energy increasing exponentially slowly from zero. The free energy will not be identically zero in the larger space of perturbed Hamiltonians for the ESM, and the magnetizations in the two phases will be expressible as left and right derivatives of the free energy with respect to an external magnetic field in this space.

One characteristic feature of the evolution observed in the simulations of the NEC presentation is that the typical excitations of one phase are triangular islands of the other phase, of the form

$$\{(i, j) \in \mathbb{Z}^2 \mid i \geqslant a, j \geqslant b, i+j \leqslant c\} \tag{5.17}$$

In the deterministic model, the southern and western boundaries of these islands are stationary, and the northeast boundary moves southwest with unit speed, so that the island disappears in a time proportional to its linear size. As noise is introduced, the southern and western boundaries acquire a drift to the south and west, respectively, and the speed of the northeast boundary decreases; islands shrink more slowly. Suppose that we continue to increase the noise and that a bias is present, say favoring $+1$ spins. Then -1 islands in the $+1$ phase will continue to shrink and disappear, but the transition to a one-phase region will be signaled by a change in the behavior of $+1$ islands in the -1 phase: sufficiently large islands of this type will now grow. In the next section we present a mean field model which captures some of these features.

5.3. Mean Field Approximations to the BG Model

In this section we discuss mean field approximations to the dynamics of the BG model. We continue to work in the NEC presentation.

Bennett and Grinstein[19] propose a dynamic mean field approximation in which sites at time $n \in \mathbb{Z}$ are assumed to have a Bernoulli distribution with magnetization m_n; that is, each spin is independently distributed with $\text{Prob}[\sigma_{n,i} = +1] = (1+m_n)/2$. The dynamics is then given by the deterministic rule

$$\begin{aligned} m_{n+1} &= -1 + p(1-\rho_n) + (1-q)(1+\rho_n) \\ &= (p-q) + \tfrac{1}{2}(1-p-q)\, m_n(3-m_n^2) \end{aligned} \tag{5.18}$$

where $(1 \pm \rho_n) = [(1 \pm m_n)^3 + 3(1 \pm m_n)^2\,(1 \mp m_n)]/4$ is twice the probability that a majority of spins in the NEC neighborhood of a site are ± 1. Equation (5.18) corresponds to a CA in $d=0$ with continuous spin variable $m \in [-1, 1]$. If we look for stationary solutions $m_n \equiv m$ of this dynamics,we find two distinct regimes: for p, q small there is a two-phase region in which there exist three stationary solutions $m_- < m_0 < m_+$, with $m_\pm$ stable and m_0 unstable; for larger p, q there is one-phase region corresponding to a single stationary solution. The phase transition curve is easily calculated explicitly, and the resulting phase plane is qualitatively similar to the true phase plane as determined by simulation (see Fig. 4).

In order to model the evolution of triangular excitations as observed in the simulations, we introduce a mean field model which can accommodate interfaces between distinct phases. Consider first a vertical interface. We suppose that the spins within the kth column at time n are independently distributed with magnetization $m_{n,k}$, and replace (5.18) by

$$\begin{aligned} m_{n+1,k} &= -1 + p(1-\rho_{n,k}) + (1-q)(1+\rho_{n,k}) \\ &= (p-q) + \tfrac{1}{2}(1-p-q)[2m_{n,k} + m_{n,k+1}(1-m_{n,k}^2)] \end{aligned} \tag{5.19}$$

where

$$\begin{aligned} (1 \pm \rho_{n,k}) = \tfrac{1}{4}[&(1 \pm m_{n,k})^2\,(1 \pm m_{n,k+1}) \\ &+ 2(1 \mp m_{n,k})(1 \pm m_{n,k})(1 \pm m_{n,k+1}) \\ &+ (1 \mp m_{n,k+1})(1 \pm m_{n,k})^2] \end{aligned} \tag{5.20}$$

is again twice the probability of a majority of ± 1 spins in the NEC neighborhood. This is now a CA in $d=1$. Clearly (5.19) has the same stationary solutions $m_{n,k} \equiv m$ as (5.18). Computationally, when p, q are in the two-phase region of the uniform mean field model, we also observe solutions corresponding to interfaces which move to the left (west):

$$m_{n,k} = f_{\pm}(k + v_{\pm} n) \tag{5.21}$$

where f_τ, $\tau = \pm 1$, is a monotonic function satisfying $\lim_{x \to \pm\infty} f_\tau(x) = m_{\pm\tau}$, and $v_\tau = v_\tau(p, q)$ is the interface velocity, observed to satisfy $0 \leqslant v_\tau < 1$.

The analysis of horizontal interfaces (which move south) is the same. For diagonal interfaces we assume that the magnetization at site $(i, j) \in \mathbb{Z}^2$ at time n is $\hat{m}_{n,i+j}$. The dynamics of $\hat{m}$ is easily written down and is similar to (5.19). Moreover, $m_{n,k} \equiv \hat{m}_{n,-(n+k)}$ obeys (5.19), so that since a vertical interface with an m_+ state on the east moves with velocity v_+, a diagonal interface with an m_- state on the northeast moves southwest with velocity $1 - v_+$.

Now consider a triangular island of the m_+ state within the m_- state. If we ignore effects due to the corners of the triangle and consider the sides as interfaces which move as above, we see that the triangle will shrink and eventually disappear if $v_+ < 1/3$, but will grow if $v_+ > 1/3$. Thus, $v_\pm < 1/3$ is the condition for stability of the $m_\mp$ phase in this model. This condition modifies the phase diagram for the mean field model by shifting the transition line between the one- and two-phase regions toward lower noise, when there is significant bias toward plus or minus states. This shift has the effect of significantly improving the agreement with the true phase diagram (see Fig. 4).

6. REMARKS AND OPEN PROBLEMS

6.1. Relation of PCA to ESM with Quenched Disorder

We consider here a PCA parametrized as in (1.6), that is, so that

$$\sigma_{n,i} = M_{\alpha,i}(\underline{\sigma}_{n-1}) \qquad \text{with probability } q_\alpha, \qquad \alpha = 1,\dots, m \tag{6.1}$$

Let us introduce independent and identically distributed random variables $\{\omega_{n,i}\}$ taking m distinct values with probabilities q_α; we might choose, for example,

$$\operatorname{Prob}[\omega_{n,i} = \alpha] = q_\alpha, \qquad \alpha = 1,\dots, m \tag{6.2}$$

Then the time evolution (6.1) may be written

$$\sigma_{n,i} = \sum_{\alpha=1}^{m} \delta_{\alpha,\omega_{n,i}} M_{\alpha,i}(\underline{\sigma}_{n-1}) \tag{6.3}$$

Consider now the random Hamiltonian on $\mathbb{Z}^{d+1}$

$$\mathscr{H}(\underline{\sigma}, \underline{\omega}) = -\sum_{n,i} \sum_{\alpha=1}^{m} \delta_{\alpha,\omega_{n,i}} \sigma_{n,i} M_{\alpha,i}(\underline{\sigma}_{n-1}) \tag{6.4}$$

For a fixed sample point $\underline{\omega}$ and initial condition σ_{-N} the energy (6.4) in the domain $\bigcup_{n>-N} \mathbb{Z}_n^d$ intuitively is minimized by choosing $\underline{\sigma}$ to be the solution of (6.3). More precisely, this is true in any domain Λ of the special type mentioned in the introduction (see ref. 6) with boundary condition σ_{-N} on the top and free boundary condition elsewhere; in fact, the ground-state energy in such a region is just $-|\Lambda|$. That is, the ground state of the ESM Hamiltonian $\mathscr{H}(\cdot, \underline{\omega})$ is a CA trajectory. We thus establish a correspondence between the PCA and a quenched random ESM at zero temperature. Note, however, that this ESM is free of frustration: for any given ω the ground state with suitable boundary conditions is unique and minimizes each single-site Hamiltonian separately.

Example 6.1. *Majority rule in one dimenson.* The model of (1.14) may be written in the form (1.6) as

$$h_0(\underline{\eta}) = (1-2\varepsilon)\, M_0(\underline{\eta}) + \varepsilon M_1(\underline{\eta}) + \varepsilon M_{-1}(\underline{\eta}) \tag{6.5}$$

where $M_0(\underline{\eta}) = \operatorname{sgn}(\eta_{-1} + \eta_0 + \eta_1)$ and $M_{\pm 1}(\underline{\eta}) = \pm 1$, leading to the random Hamiltonian

$$\mathscr{H}(\underline{\sigma}, \underline{b}) = -\sum_{n,i} [(1-|b_{n,i}|)\, \sigma_{n,i} \operatorname{sgn}(\sigma_{n-1,i-1} + \sigma_{n-1,i} + \sigma_{n-1,i+1}) - b_{n,i}\sigma_{n,i}] \tag{6.6}$$

where now

$$b_{n,i} = \begin{cases} 1 & \text{with probability } \varepsilon \\ -1 & \text{with probability } \varepsilon \\ 0 & \text{with probability } 1-2\varepsilon \end{cases} \tag{6.7}$$

The entropy per site associated with the randomness in the Hamiltonian (6.4) is given by

$$\tilde{s}(\underline{q}) = -\sum_{\alpha} q_\alpha \log q_\alpha \tag{6.8}$$

In general, there are many possible decompositions (1.6) for a given PCA, and the infimum of $\tilde{s}(\underline{q})$ over all such decompositions is the minimum possible entropy for an equivalent random Hamiltonian. At zero temperature $\tilde{s}$ is the entropy per site for the joint distribution of the variables $\underline{\sigma}$ and $\underline{\omega}$, and hence is an upper bound for the entropy per site $\hat{s}(\mu_\nu)$ of the ESM in state μ_ν [see (1.26)]. In fact, the entropy (6.8) is greater than or equal to the PCA noise (1.25). To see this, we define for given $\underline{\eta}$

$$q_\pm = \sum_{\alpha \in Q_\pm(\underline{\eta})} q_\alpha \tag{6.9}$$

where $Q_\pm(\underline{\eta}) = \{\alpha \mid M_\alpha(\underline{\eta}) = \pm 1\}$, so that $s_0(\underline{\eta}) = -q_+ \log q_+ - q_- \log q_-$, and then observe by the convexity of the logarithm that

$$-\sum_{\alpha \in Q_\pm} \frac{q_\alpha}{q_\pm} \log q_\alpha \geqslant -\log\left(\sum_{\alpha \in Q_\pm} \frac{q_\alpha^2}{q_\pm}\right) \geqslant -\log q_\pm \tag{6.10}$$

so that $\tilde{s}(\underline{q}) \geqslant s_0(\underline{\eta})$ for all $\underline{\eta}$.

There are other ways to identify PCA with random ESM. For example, the finite-temperature Gibbs measure

$$\mu_{\beta, \underline{b}}(d\underline{\sigma}) = Z^{-1} \exp[-\beta \mathscr{H}(\underline{\sigma}, \underline{b})]\, \rho_0(d\underline{\sigma}) \tag{6.11}$$

where ρ_0 is the *a priori* Bernoulli measure giving all spin configurations equal weight and $\mathscr{H}(\underline{\sigma}, \underline{b})$ is given by (6.6), corresponds to a PCA in which the evolution (6.1) is followed with probability $1 - 2\varepsilon$ and a random choice of ± 1 is made with probability 2ε, where $\varepsilon = (e^{2\beta} + 1)^{-1}$. More generally, if h is a convex combination $h_i = \sum_{\alpha=1}^m q_\alpha h_i^\alpha$ of h-functions for other PCA, then we may think of the ESM as determined by a random Hamiltonian in which the Hamiltonian H_x^α corresponding to h^α acts at site x with probability q_α, all choices being independent. These identifications may make it possible to bring the methods of random ESM to bear on the study of PCAs.

6.2. Stability of CA Trajectories

We consider here PCAs which are nearly deterministic, that is, which satisfy

$$\|h_0 - M_0\| < 2\varepsilon \tag{6.12}$$

for some $M_0(\underline{\eta})$ with $|M_0(\underline{\eta})| = 1$ and some small $\varepsilon > 0$. We will call a trajectory $\underline{\sigma}$ for the deterministic CA determined by M_0 *stable* if PCA histories which agree with $\underline{\sigma}$ at some initial time remain close for all time;

specifically, if for any $\delta > 0$ there is an $\varepsilon > 0$ such that if $\tilde{\underline{\sigma}}$ is a history for a PCA satisfying (6.12), with $\underline{\sigma}_{-N} = \tilde{\underline{\sigma}}_{-N}$, then $\text{Prob}[\tilde{\sigma}_{n,i} \neq \sigma_{n,i}] < \delta$ for all n, i with $n > -N$. If at least two trajectories for the CA are stable, then the PCA must have two low-noise phases, so that it is natural to base a low-noise analysis of PCA on the study of the CA trajectories. The corresponding study of low-temperature phases of an ESM as perturbations of ground states gives rise to the Pirogov–Sinai theory.[43] There is a direct connection between these ideas: for the simplest perturbation [see (1.28)],

$$h_0(\underline{\eta}) = M_0(\underline{\eta})(1 - 2\varepsilon) \tag{6.13}$$

the trajectories of the CA are ground states for the corresponding ESM, which has [see (1.14)],

$$H_0(\sigma_0, \eta) = -\beta\sigma_0 M_0(\eta) - \text{const}, \qquad \beta = \frac{1}{2}\log\frac{1-\varepsilon}{\varepsilon} \tag{6.14}$$

In general, when (6.12) is satisfied with ε sufficiently small, the same conclusion holds in finite volumes whose maximum size may depend on ε.

Stability of CA trajectories is closely related to their stability against finite perturbations (see Section 5), which we will call *weak stability*. Specifically, a CA trajectory $\underline{\sigma}$ is weakly stable if, whenever $\hat{\underline{\sigma}}$ is another trajectory which agrees with $\underline{\sigma}$ at some time $-N$ except at a finite number of sites, then $\underline{\sigma}$ and $\hat{\underline{\sigma}}$ are identical after some time $N' \geqslant -N$. Toom[9] has shown that for attractive CAs (see Section 3) in which the function M_0 is not constant, weak stability of one of the trajectories $\underline{\sigma}^{\pm}$ defined by $\sigma^{\pm}_{n,i} \equiv \pm 1$ implies its stability in any PCA satisfying (6.12) with sufficiently small ε. (Toom uses the term attractivity for what we have called weak stability.) Moreover, he has proved a partial converse: under the same hypotheses, lack of weak stability for one of $\underline{\sigma}^{\pm}$ implies that, for any $\varepsilon > 0$, that trajectory is not stable for *some* PCA satisfying (6.12).

In the ESM picture, in which a CA trajectory is the ground state of an appropriate finite-range Hamiltonian, weak stability implies that the energy of an excitation over a region Λ goes to infinity as $|\Lambda|$ does. This is weaker than the *Peierls condition*, which plays a central role in the Pirogov–Sinai theory and requires that inserting a ground state in a volume Λ into another ground state costs an energy proportional to the size of the boundary of Λ. Weak stability is more closely related to the condition of *regularity* as introduced by Slawny[44] and Bricmont and Slawny[45]: an ESM is regular if the energy of an excitation relative to the corresponding ground state tends to infinity with the size of the excitation. It is not clear at present, however, how to apply these low-temperature theories for the ESM to the problem of phase transitions in the PCA.

The simplest candidates for low-temperature phases of a PCA are states with a preponderance of spins of one sign; by Toom's theorem, the existence of such phases for perturbations of attractive CA follows from the weak stability of the CA trajectories $\underline{\sigma}^{\pm}$. Moreover, by the partial converse, results on the lack of weak stability for these trajectories in one dimension thus give some support to the positive rates conjecture. We give two such results.

The first of these is a simple observation. We say that a $d=1$ CA *preserves interfaces* if its evolution leaves invariant the configurations $\pi^{\pm}$ defined by

$$\pi_i^{\pm} = \begin{cases} \pm 1 & \text{if} \quad i \geqslant 0 \\ \mp 1 & \text{if} \quad i < 0 \end{cases} \tag{6.15}$$

Then the configurations $\underline{\sigma}^{\pm}$ cannot be weakly stable for a $d=1$ finite-range CA which preserves interfaces, since the insertion of a sufficiently large interval of spins of the opposite sign will necessarily lead to an invariant configuration.

The second result follows from a criterion for weak stability due to Toom. We call a set $A \subset \mathbb{Z}_{-1}^d$ a *plus set* if having $\eta_i = 1$ for all $i \in A$ ensures that $M_0(\underline{\eta}) = 1$; A is *minimal* if it contains no other plus sets. Minimal minus sets are defined similarly. We now identify $\mathbb{Z}_{-1}^d$ with $\mathbb{Z}^d \subset \mathbb{R}^d$ in the natural way and for any $A \subset \mathbb{Z}_{-1}^d$ we write $\mathrm{Conv}(A)$ for the convex hull of A in $\mathbb{R}^d$. Then Toom proves the following criteria: for attractive CA, the trajectory $\underline{\sigma}^{+}$ (respectively $\underline{\sigma}^{-}$) is weakly stable if and only if

$$\bigcap \mathrm{Conv}(A) = \varnothing \tag{6.16}$$

where the intersection is over all minimal plus (respectively minus) sets A. Note that each plus set A must intersect each minus set B, since otherwise a contradiction would follow from consideration of a configuration $\underline{\eta}$ with $\eta_i = +1$ on A, $\eta_i = -1$ on B.

Example 6.2. For the deterministic one-dimensional majority vote model (Example 1.1 with $\varepsilon = 0$), or for the deterministic Toom model, a minimal plus set or a minimal minus set consists of any two of the three points in $\mathbb{Z}_{-1}^d$ which determine $\sigma_{0,0}$. The sets $\mathrm{Conv}(A)$ are then line segments joining the pairs of points. Condition (6.16) clearly holds in the second case and not in the first.

Toom's criterion implies immediately our second negative result on weak stability in one dimension:

Proposition. There does not exist an attractive CA in dimension $d=1$ for which both $\underline{\sigma}^{+}$ and $\underline{\sigma}^{-}$ are weakly stable.

Proof. If $\underline{\sigma}^+$ is weakly stable, then there must exist two minimal plus sets, say A_1 and A_2, for which $\mathrm{Conv}(A_1)$ and $\mathrm{Conv}(A_2)$ are separated by some open interval I. But all minimal minus sets must intersect both A_1 and A_2 and hence their convex hulls must contain I, so that $\underline{\sigma}^-$ is not weakly stable. ▮

6.3. Nonlocal Updating

The PCA–ESM correspondence extends to those PCAs for which the function h_0 appearing in (1.3) is not of finite range. It suffices to require, for example, that the variation $\Delta_{h_0}(i)$ [see (4.11)] be exponentially small in the distance $|i|$; the corresponding ESM will then have an exponentially decreasing interaction potential. Aside from its intrinsic interest, this generalization would be relevant for the extension of our discussion to Markov processes which are discretizations of continuous-time processes (interacting particle systems) obtained by observing the processes only at integer times. In the continuous-time system (except in trivial cases), and hence in discrete time, the distribution of the spin at site (t, i) will always depend on the spin configuration at time $t = 0$ on all of $\mathbb{Z}^d$. Chapter 1 of Liggett[13] gives estimates on this dependence which yield the desired exponential decrease in the discretized model.

The correspondence also extends to PCA in which the updating at time n depends on spins at all times previous to n, or, more generally, in which the updating of $\sigma_{n,i}$ depends on all spins in the lexicographic past of (n, i). Again, some exponential decrease in the interaction strength is required. See ref. 6 for further discussion. A potentially interesting example of a PCA depending on a lexicographic past would be obtained by adding noise to the *parity rule filter automata*,[46] a CA which has been proved to exhibit solitonlike behavior.[47]

6.4. More Inequalities

As observed in Section 3, positive dynamics for a PCA does not give rise to ferromagnetic couplings in the ESM [there are some trivial exceptions when $\log Q(\underline{\eta})$ is independent of $\underline{\eta}$]. Correlations are nevertheless positive because we may reorganize the PCA measure into the form (3.6). We now observe that one may write any Gibbs measure in a volume Λ, not just those for ESM corresponding to PCAs, in the form

$$\mu_\Lambda(d\sigma) = K_\Lambda \prod_{x \in \Lambda} \prod_a \left(1 + \sum_{B \subset \Lambda} p_{B,x}^{(a)} \sigma_B\right) \rho_0(d\sigma) \tag{6.17}$$

where K_Λ is a normalization constant and ρ_0 is the *a priori* measure giving all spin configurations equal weight, and there is no restriction on the sets B in the sum; μ_Λ must of course be nonnegative. It is clear that we will have positive correlations when all $p_{B,x}^{(a)}$ are nonnegative. This construction is related to that of Fernandez and Slawny[28] in their extension of GKS inequalities to systems with some nonferromagnetic interactions. The connection is via the remark after (1.20) that antiferromagnetic terms from $\log[2\cosh Q_0(\underline{\eta})]$ contain only bonds which are symmetric differences of bonds in the ferromagnetic $Q_0(\underline{\eta})$.

To illustrate the above construction, let us again consider the example discussed immediately after (5.4) on the lattice of Fig. 1. This ESM may be thought of as an Ising system on a square lattice $\mathbb{Z}^2$ with nearest-neighbor ferromagnetic coupling J and next-nearest-neighbor antiferromagnetic coupling $-J' = -\frac{1}{2}\log\cosh 2J$ between spins at sites $(i, j+1)$ and $(i+1, j)$. [In the PCA language, time is now increasing along diagonals running from $(i+1, j+1)$ to (i, j).] We now introduce a measure $\tilde{\mu}_\Lambda$ in a periodic square box Λ, of the form

$$\tilde{\mu}_\Lambda = K_\Lambda \prod_{(i,j)\in\Lambda} \tfrac{1}{2}F_{ij}^1 F_{ij}^2 \rho(d\sigma) \tag{6.18}$$

where

$$\begin{aligned} F_{ij}^1(\underline{\sigma}) &= \prod_{a=1}^{n_1} \left[1 + \tfrac{1}{2}\tanh(2p^{(1,a)}J)\,\sigma_{ij}(\sigma_{i,j+1} + \sigma_{i+1,j})\right] \\ F_{ij}^2(\underline{\sigma}) &= \prod_{a=1}^{n_2} \left[1 + \tfrac{1}{2}\tanh(2p^{(2,a)}J)\,\sigma_{ij}(\sigma_{i,j+1} + \sigma_{i-1,j})\right] \end{aligned} \tag{6.19}$$

with $\sum_{a=1}^{n_1} p^{(1,a)} + \sum_{a=1}^{n_2} p^{(2,a)} = 1$ and all $p^{(i,a)}$ nonnegative. The corresponding ESM, which of course does not correspond to a PCA, still has nearest-neighbor couplings J, but now has next-nearest-neighbor couplings $-J_1'$ and $-J_2'$ on the $(i, j+1)$–$(i+1, j)$ and $(i, j+1)$–$(i-1, j)$ diagonals, respectively, where

$$J_i' = \sum_{a=1}^{n_i} \tfrac{1}{2}\log\cosh 2p^{(i,a)}J \tag{6.20}$$

It is easy to verify that by choosing the n_i and $p^{(i,a)}$ appropriately one may obtain any diagonal antiferromagnetic couplings J_1', J_2' satisfying

$$\cosh^{-1}(\exp 2J_1') + \cosh^{-1}(\exp 2J_2') \leqslant 2J \tag{6.21}$$

By the above argument, all of these ESM have positive correlations.

6.5. Linear Dynamics

It has been remarked by various authors (see refs. 7 and 8 and references therein) that the PCA and corresponding ESM are "exactly solvable" when $h_0(\underline{\eta})$ has the form

$$h_0(\underline{\eta}) = \tilde{r}_0 + \sum_{i \in U} \tilde{r}_i \eta_i \tag{6.22}$$

such PCA are called *affine*, or *linear* if $\tilde{r}_0 = 0$. In this case the inequality in (3.4) becomes an equality and one can solve explicitly for the correlation functions of the ESM.[8] In particular, there will be no phase transition when $|h_0(\underline{\eta})| < 1$, since for (6.22) this condition implies $|\tilde{r}_0| + \sum |\tilde{r}_i| < 1$, so that (4.1) is always satisfied. Note that if $Q_0(\underline{\eta}) = -J_0 - J_1\eta_1$ or $Q_0(\underline{\eta}) = -J_1\eta_1 - J_2\eta_2$, then it follows from (1.22) that the PCA is affine or linear, respectively.

6.6. Open Problems

In this section we summarize several open problems involving the PCA–ESM correspondence, including some which have been mentioned earlier.

1. We expect that increasing the noise in a PCA will decrease its ability to remember its initial configuration, but in exactly what sense this is true is not clear at present. To formulate a precise question, we consider a PCA of the form

$$h_0(\underline{\eta}) = (1 - 2\varepsilon)\, M_0(\underline{\eta}) \tag{6.23}$$

where $0 \leqslant \varepsilon \leqslant 1/2$, M_0 is an increasing function (so that the PCA is attractive), and $|M_0| = 1$. Examples are the majority vote model of (1.13) and the BG model with $p = q = \varepsilon$. This simple PCA follows the deterministic rule determined by M_0 with probability $1 - 2\varepsilon$ and makes an independent choice of $+1$ or -1 with probability ε for each; ε is thus a measure of the noise. Suppose that the PCA is started at time $-N$ with the configuration in which all spins are $+1$, that is, in the initial measure $\rho = \delta_+$ discussed at the end of Section 3, and that C is a subset of $\mathbb{Z}^{d+1}$ in the future of $-N$ with $\langle \sigma_C \rangle$ its expectation in the measure μ_ρ^{-N}. We ask: Does $\langle \sigma_C \rangle \geqslant 0$ hold, as suggested by attractivity? Is $\langle \sigma_C \rangle$ a decreasing function of ε?

The same questions could be posed for the model of Example 5.2 or that discussed in connection with Fig. 1, in which β is a measure of the noise. Here we know that $\nu = \lim_{n \to \infty} \delta_+ P^n$ is the measure for an Ising model and has positive correlations, but we cannot show that the equal-time correlations at finite times are positive.

2. Consider an ESM arising from a PCA, i.e., with Hamiltonian given by (1.10)–(1.11). Are all Gibbs measures Markov chain measures for the PCA? The question has been answered affirmatively for translation-invariant and periodic Gibbs states in ref. 6.

3. Let ν be a measure on $\mathbb{Z}^d$ which is invariant for some PCA. Under what conditions is ν Gibbsian? Sufficient conditions are given in Section 4; basically, it suffices that the corresponding ESM exist and be in the high-temperature Dobrushin–Shlosman regime in which the Gibbs state is unique. In the low-temperature regime, on the other hand, the non-Gibbsian nature of ν has been established for certain dynamics (see refs. 25 and 26 for the voter model and refs. 48 and 49 for a related question). The question remains open for the regime in which one of the high-noise conditions (4.1) or (4.3) of Chapter 4, but not the Dobrushin–Shlosman condition, is satisfied, and for most models at low or intermediate temperature. We remark that in the continuous-time case, in which there is no correspondence with an ESM, the Gibbsian nature of the stationary measure ν is entirely open.

4. Dobrushin and Shlosman have constructed a family of conditions C_V on a general ESM, where $V \subset \mathbb{Z}^{d+1}$ is any finite volume, such that the Gibbs state is unique if and only if C_V holds for some volume V. The criterion is in principle able to locate the phase-transition threshold with arbitrary accuracy. C_V is satisfied if the Gibbs measure in V is only weakly influenced by the variation of boundary conditions, that is, of spins outside V. We observed in Section 4 that our Condition 2 was a less restrictive form of the criterion $C_{\{0\}}$—less restrictive because we need consider only variations of the spin configuration in the past of 0. We ask: Can one find similarly less restrictive conditions C'_V which characterize completely the uniqueness region for the PCA, but which involve only the dependence on spins in the past of V, that is, which are expressed in terms of the PCA dynamics?

APPENDIX A. TOOM'S THEOREM

In this Appendix we discuss the Toom model in the symmetric presentation, that is, on the lattice $\mathbb{L} = \mathbb{Z}^3$ with $t(x) = -(x_1 + x_2 + x_3)$. We consider a PCA with transition probabilities satisfying (5.11), so that, for any site $x \in \mathbb{L}$,

$$\mathrm{Prob}[\sigma_x \neq \phi_x(\underline{\sigma}_{t(x)-1})] < \varepsilon \tag{A.1}$$

Let δ_+ be the single-time measure concentrated on the configuration $\sigma_x \equiv +1$, let $\mu^{-N} \equiv \mu_{\delta_+}^{-N}$ be the space-time measure on $\{x \in \mathbb{L} \mid t(x) \geqslant -N\}$

generated by taking δ_+ as initial condition at time $-N$, and let ν be the invariant measure $\nu = \lim_{n\to\infty} \delta_+ P^n$. Our goal is to prove Toom's theorem on the existence of two stationary states.

Theorem A.1. For sufficiently small $\varepsilon > 0$ there exists a constant $c < 1/2$ such that

$$\mu^{-N}(\{\sigma_0 = -1\}) \leqslant c \tag{A.2}$$

for all $N > 0$, and hence $\nu(\{\sigma_0 = -1\}) \leqslant c$. The same conclusion holds if $+1$ and -1 are interchanged.

We discuss the proof in several stages, moving from its general structure as a Peierls argument to specific details of the contours (here graphs) which are needed. We remark also that a new and quite simple proof of the result has recently been given by Gray and Bramson.[50]

A1. The Peierls Argument

In the variant of the Peierls argument used by Toom, the role normally played by contours is taken by certain *graphs*. Note that although the theorem may be regarded as asserting the existence of a phase transition in a three-dimensional Hamiltonian lattice system, the "contours" are nevertheless one-dimensional objects. In this section we give a minimal description of the graphs involved, specifying only the information needed to understand how the basic Peierls estimate works.

Let $\underline{\sigma}$ be a configuration with $\sigma_0 = -1$. We may (and usually will) specify this configuration by giving the set $X \equiv X(\underline{\sigma}) \equiv \{x \in \mathbb{L} \mid \sigma_x = -1\}$. Define

$$\hat{X} = \{x \in X \mid \sigma_x \neq \phi_x(\underline{\sigma}_{t(x)-1})\} \tag{A.3}$$

Sites in $\hat{X}$ are those sites of X at which we know that the deterministic rule has not been followed; we say that an *error* has occurred and call the sites in $\hat{X}$ *error sites.*

Given X (or equivalently $\underline{\sigma}$), we will shortly define a (finite) graph $G = G(X)$. [The rules we will give do not specify $G(X)$ uniquely, but we will assume that appropriate choices have been made for each X so that $G(X)$ is well defined.] G has vertex set $V_G \subset X$, edge set E_G, and distinguished vertex subset $\hat{V}_G = V_G \cap \hat{X}$. These graphs satisfy two key properties, uniformly in N. First, the number of possible graphs grows at worst exponentially in the number of edges:

$$|\{G \mid |E_G| = m\}| \leqslant (48)^{2m} \tag{A.4}$$

Second, the number of edges in a graph cannot be too large compared to the number of vertices which are error sites:

$$|\hat{V}_G| = |V_G \cap \hat{X}| \geqslant \tfrac{1}{4}|E_G| + 1 \tag{A.5}$$

If we assume that graphs can be defined satisfying these conditions, then the theorem follows immediately from (A.1) and the independence of updating at each site:

$$\begin{aligned}
\operatorname{Prob}[\sigma_0 = -1] &= \sum_{m=0}^{\infty} \sum_{\{G \mid |E_G| = m\}} \operatorname{Prob}[G(X(\underline{\sigma})) = G] \\
&\leqslant \sum_m \sum_G \operatorname{Prob}[\text{An error occurs at each } x \in \hat{V}_G] \\
&\leqslant \sum_m (48)^{2m} \varepsilon^{(m/4)+1}
\end{aligned} \tag{A.6}$$

and this sum is strictly less than 1/2 for ε sufficiently small.

A2. The Toom Graphs

In this section we describe further the graphs introduced above, by listing a series of seven properties which they possess. The construction of the graphs will be described in the next section. We need some preliminary notation.

For $x \in X$ we let $\tilde{U}(x)$ be the set of sites directly responsible under the deterministic rule for the fact that $\sigma_x = -1$; of course, no site is responsible if x is an error site:

$$\tilde{U}(x) = \begin{cases} \varnothing & \text{if} \quad x \in \hat{X} \\ [U_0 + x] \cap X & \text{otherwise} \end{cases} \tag{A.7}$$

Further, for $A \subset X$ we let $\tilde{U}(A)$ be the set of sites directly responsible for A, and $W(A)$ be the set of all sites directly or indirectly responsible for A:

$$\begin{aligned}
\tilde{U}^0(A) &= A \\
\tilde{U}^1(A) \equiv \tilde{U}(A) &= \bigcup_{x \in A} \tilde{U}(x) \\
\tilde{U}^{(n)}(A) &= \tilde{U}(\tilde{U}^{(n-1)}(A)) \\
W(A) &= \bigcup_{n \geqslant 0} \tilde{U}^{(n)}(A)
\end{aligned} \tag{A.8}$$

We write $Y = W(\{0\})$ and $\hat{Y} = Y \cap \hat{X}$, and will construct G so that $V_G \subset Y$.

The characterizing properties of the Toom graphs are as follows.

P1. Each graph $G = G(X)$ is connected, and the origin belongs to V_G.

P2. There are two types of edges in G. *Timelike* edges join two nearest-neighbor sites in $\mathbb{L}$; the times associated with these sites necessarily differ by one. *Spacelike* edges join nearest-neighbor sites within a fixed-time triangular lattice $\mathbb{L}_n$; these are next-nearest-neighbor sites in $\mathbb{L}$. (Toom calls timelike and spacelike edges *arrows* and *forks*, respectively.) More than one timelike edge can join the same pair of sites.

P3. The edges of G are oriented, so that we may define an incidence matrix α: for $x \in V_G$ and $e \in E_G$,

$$\alpha_{xe} = \begin{cases} 1 & \text{if } \; e \text{ is oriented into } x \\ -1 & \text{if } \; e \text{ is oriented out of } x \\ 0 & \text{otherwise} \end{cases} \tag{A.9}$$

P4. Each edge e carries an *index* $k(e)$ equal to 1, 2, or 3, and a corresponding *current* $j_e \in \mathbb{Z}^3$, depending only on $k(e)$:

$$j_e = \begin{cases} (2, -1, -1) & \text{if } \; k(e) = 1 \\ (-1, 2, -1) & \text{if } \; k(e) = 2 \\ (-1, -1, 2) & \text{if } \; k(e) = 3 \end{cases} \tag{A.10}$$

P5. Timelike edges are restricted in two ways: they are always oriented in the direction of decreasing time, and the index $k = k(e)$ must be such that the kth component of the lattice site does not change along e. That is, if in general we write $\delta_e = \sum_x x\alpha_{xe}$ for the displacement along edge e, then $\delta_e \cdot j_e = 1$ for timelike edges.

P6. Current is conserved at each vertex $x \in V_G$: $\sum_e \alpha_{xe} j_e = 0$. Geometrically, this means that the edges incident on x may be sorted into *pairs* of edges with the same index, one entering and one leaving x, and *triplets* of edges with distinct indices, all entering or all leaving x.

P7. The number s of spacelike edges in G satisfies $s = |\hat{V}_G| - 1$.

Properties P1–P7 imply (A.4) and (A.5). To verify (A.4), note that in any connected graph G there exists a closed walk which traverses each edge precisely twice (this is just an Euler path in a new graph obtained by doubling each edge of G). We may estimate the total number of graphs by estimating the number of walks consistent with P1–P7. Begin the walk at the origin and note that at each step there are at most 48 choices: we may go in any of six spacelike directions, and for each direction choose between two orientations and among three indices, or we may go in any of six timelike directions, and for each direction choose between two indices. Thus, the total number of walks for graphs with $|E_G| = m$ is at most $(48)^{2m}$.

Equation (A.5) follows from current conservation (P6): since $\delta_e \cdot j_e \geqslant -3$ for e spacelike,

$$\begin{aligned}
0 &= \sum_{x \in V_G} \sum_{e \in E_G} x \cdot (\alpha_{xe} j_e) \\
&= \sum_{e \in E_G} \delta_e \cdot j_e \\
&\geqslant \sum_{e \text{ timelike}} (1) + \sum_{e \text{ spacelike}} (-3) \\
&\geqslant |E_G| - 4s \\
&= |E_G| - 4(|\hat{V}_G| - 1) \qquad \text{(A.11)}
\end{aligned}$$

A3. Construction of Graphs

In this section we construct, for each configuration X, a graph G satisfying P1–P7 above.

The first step is to partition Y into equivalence classes which we call *clusters*. For $n \geqslant 0$ let $Y_n = \{x \in Y \mid t(x) = -n\}$; each cluster is wholly contained in some Y_n. The equivalence relation on Y_n is generated by an adjacency relation: for $x, y \in Y_n$,

$$x, y \text{ adjacent} \Leftrightarrow W(x) \cap W(y) \neq \varnothing \qquad \text{(A.12)}$$

If we regard Y_n as a graph under this relation, then the clusters are simply the connected components of the graph; more explicitly, x and y belong to the same cluster if and only if there is a sequence $x = x_0, x_1, \ldots, x_k = y$ of sites in Y_n with $W(x_k) \cap W(x_{k+1}) \neq \varnothing$. Note that $\{0\}$ is the unique cluster in Y_0.

A cluster $B \subset Y_{n+1}$ will be called a *parent* of a cluster $A \subset Y_n$ if $B \cap \tilde{U}(A) \neq \varnothing$. We claim that no such B can be the parent of distinct clusters A and A'. Otherwise, there would exist $a \in A$, $a' \in A'$, and $b, b' \in B$ with $b \in \tilde{U}(a)$ and $b' \in \tilde{U}(a')$, and a chain $b = b_1, b_2, \ldots, b_m = b'$ with $b_i \in B$. We may suppose that (for given B) the clusters A, A' and the elements $b_1, \ldots, b_m$ are chosen to make m as small as possible. Because $b_{m-1} \in Y = W(0)$, however, there must exist an $a'' \in Y_n$ with $b_{m-1} \in \tilde{U}(a'')$; since $a'' \in A''$ for some cluster $A'' \subset Y_n$, replacing A' by A'', a' by a'', and b' by b_{m-1} leads to a chain with a smaller value of m.

For a cluster $A \subset Y_n$ we define a graph $H(A)$ whose vertices are the parents of A; edges in $H(A)$ will be called *links* to distinguish them from edges in G. Parents B and B' are joined by a link in $H(A)$ if $\tilde{U}(a) \cap B \neq \varnothing$

and $\tilde{U}(a) \cap B' \neq \varnothing$ for some $a \in A$. The fact that A is a cluster implies that $H(A)$ is connected. For, if B and B' are parents of A, there exist $a, a' \in A$ with $\tilde{U}(a) \cap B \neq \varnothing$ and $\tilde{U}(a') \cap B' \neq \varnothing$, a chain $a = a_1, a_2, \dots, a_m = a'$ with $W(a_k) \cap W(a_{k+1}) \neq \varnothing$, and hence elements $b'_k \in \tilde{U}(a_k)$, $b_{k+1} \in \tilde{U}(a_{k+1})$ with $W(b'_k) \cap W(b_{k+1}) \neq \varnothing$. If B_k $(k = 1, \dots, m-1)$ is the cluster containing b'_k and b_{k+1}, and $B_0 = B$, $B_m = B'$, then for each k, B_k and B_{k+1} either coincide or are adjacent in $H(A)$; thus, there is a ath from B to B' in $H(A)$.

We are now ready to define the graph $G(X)$. We will construct G inductively, defining $G_0, G_1, \dots$ and setting $G = \bigcup_n G_n$. The graph G_n will consist of that portion of G lying at time $-n$ or later, that is, V_{G_n} is equal to $V_G \cap \bigcup_{m \leqslant n} Y_m$ and E_{G_n} consists of all edges of G for which the ends x and y both lie in $\bigcup_{m \leqslant n} Y_m$. Suppose for the moment that we have constructed G_n. If G_n intersects a cluster $A \subset Y_n$ then we will say that A is *relevant*. We further classify relevant clusters by noting that, if $x \in Y_n \cap \hat{Y}$, then $\{x\}$ is a cluster; we call $\{x\}$ *terminal*. Relevant clusters not of this form are *active*.

Each G_n will satisfy properties P2–P5 above and will satisfy modified versions of P1, P6, and P7 as follows.

P1′. The graph obtained from G_n by contracting each relevant cluster $A \subset Y_n$ to a point is connected, and the origin belongs to V_{G_n}.

P6′. Current is conserved at each vertex x of G_n for which $t(x) > -n$. Current may not be conserved at vertices in $V_{G_n} \cap Y_n$, but is conserved for clusters: for any cluster $A \subset Y_n$, $\sum_{x \in A} \sum_{e \in E_{G_n}} \alpha_{xe} j_e = 0$. In fact, for $n > 0$ we will make an even more restrictive assumption: if $A \subset Y_n$ is a cluster, then one of the following cases holds.

(i) Precisely one edge of G_n enters A, and precisely one edge, with the same index, leaves A. We call A a *biped.*

(ii) Precisely three edges of G_n, with distinct indices, enter vertices of A, and no edges leave A. We call A a *triped.*

The vertices of A on which any of these edges are incident are not necessarily distinct.

P7′. Let s_n be the number of spacelike edges in G_n, let $\hat{V}_{G_n} = V_{G_n} \cap \hat{X}$, and let c_n be the number of active clusters $A \subset Y_n$. Then

$$s_n = c_n + |\hat{V}_{G_n}| - 1 \tag{A.13}$$

We begin the induction by defining G_0 to be the trivial graph with vertex set $\{0\}$ and with no edges. P1′, P2–P5, and P6′ are trivially satisfied. To check P7′, note that $s_0 = 0$ and that $c_0 = 0$, $|\hat{V}_{G_0}| = 1$ if $\{0\}$ is terminal (i.e., lies in $\hat{Y}$), $c_0 = 1$, $|\hat{V}_{G_0}| = 0$ if $\{0\}$ is active.

Now assume inductively that we have defined G_n. If every cluster in G_n is terminal, i.e., consists of a single vertex in $\hat{Y}$, then the construction terminates with $G = G_n$. This must happen for some n less than $-N$; when it does, P1′, P6′, and P7′ become P1, P6, and P7, finishing the proof. If, alternatively, G_n contains at least one active cluster, then for each such cluster $A \subset Y_n$ we will construct, according to steps 1 and 2 below, the portion of G_{n+1} which lies above A, i.e., in $\tilde{U}(A)$. The induction step is finished when this construction has been carried out for each active cluster.

Thus, let us fix an active cluster A of G_n. We carry out our construction as follows.

Step 1. Add to G_n a total of three timelike edges, carrying distinct nonzero indices, each joining some vertex $a \in A \cap G_n$ to a vertex in $\tilde{U}(a)$, and also add the vertices of Y_{n+1} which are the endpoints of these edges. If $n = 0$, the new edges are incident on the unique vertex 0 of G_0. If $n > 0$, the new edges are incident on vertices a of A which are in V_{G_n}, that is, on vertices on which edges of G_n are already incident, and are chosen so that, in the new graph, current conservation is satisfied at each vertex of A. The procedure to do so depends on which of the alternatives assumed in P6′ holds, that is, on whether A is a biped, with legs having some index k, or a triped. In the first case we add one timelike edge, of index k, to the vertex at which the edge of G_n enters, and two edges, of distinct indices not equal to k, to the vertex which the edge of G_n leaves (these vertices may coincide). In the second case we add one edge to each vertex at which an edge of G_n enters, with the index of the entering edge.

To complete the description of this step, we must specify the displacements of the added edges, or equivalently their upper endpoints. There is some freedom in this step, and this is one source of the lack of uniqueness in the assignment of G to X. The only constraint is that of P5: if the edge has index k (k is already determined by the above prescription), then the kth component of the displacement along the edge must be zero. Suppose, however, that the edge is to be added at $a \in A$. Because A is active, $\tilde{U}(a)$ contains at least two vertices, and it is always possible to choose one of these as the endpoint of the edge while satisfying the constraint. We may assume that, when more than one choice is allowable, the ambiguity is removed by some predetermined rule.

Step 2. Now recall that $\tilde{U}(A)$ has been partitioned into clusters, called the parents of A; we want to add spacelike edges joining pairs of vertices of $\tilde{U}(A)$ so that each parent will satisfy the overall current conservation required by P6′.

Suppose that the timelike edge of index k, $k = 1, 2, 3$, which was added at step 1 is incident on a vertex of $\tilde{U}(A)$ lying in a cluster B_k; two or all three of the clusters B_1, B_2, and B_3 may coincide. Because the graph $H \equiv H(A)$, whose vertices are the parents of A, is connected, we can choose a minimal tree T in H connecting the vertices B_1, B_2, and B_3. We want to assign orientations and current indices to the links of T. Current enters T at these distinguished vertices; there is then clearly a unique way to specify an orientation and an index k for each link in T so that current conservation is satisfied at each vertex of T. Note that it is possible for B_1, B_2, and B_3 to coincide, so that T consists of a single vertex and no lines of T are assigned current.

Finally, we add spacelike edges to G_{n+1}. By the definition of adjacency in H, any two parents B and B' which are adjacent as vertices of T must both intersect some $\tilde{U}(a)$ for $a \in A$; hence, there exist $b \in B$ and $b' \in B'$ which are adjacent (i.e., next nearest neighbors) in the lattice. For each such pair B, B' we add to G_{n+1} a spacelike edge joining b and b', with the same index as, and corresponding orientation to, the link of T joining B with B'. This completes the construction of the portion of G_{n+1} lying above A.

Suppose now that we have constructed G_{n+1} by applying steps 1 and 2 to each active cluster of G_n. It is easy to see that the resulting graph posesses properties P2–P5 and P1′, P6′, and P7′; we will discuss only the last two of these. Consider what happens during the application of steps 1 and 2 to a particular active cluster A. For P6′, note that each parent of A which is a vertex of T becomes a biped or a triped in G_{n+1}, according to the sum of its degree in T and the number of timelike edges from step 1 which enter it (there will be exactly one triped). Parents which are not vertices of T are irrelevant in G_{n+1}. For P7′, note that if T contains r vertices, r' of which are terminal clusters of G_{n+1}, then we have added $r-1$ spacelike edges in step 2, increased $\hat{V}_{G_n}$ be r', and added $r-r'$ new active clusters and removed one (A itself), so that (A.13) is preserved.

This completes the proof of Toom's theorem. Figure 5 shows a simple Toom graph constructed by this procedure, Fig. 6 a more complicated example with branching. Note that current conservation forces the graphs to have a characteristic three-branched structure. ∎

APPENDIX B. DETAILED BALANCE

In this appendix we make some brief comments on the question of detailed balance for the PCA evolution with respect to its invariant measures. The condition $\nu = \nu P$ for invariance may from (1.8) be written in the form

$$\int [P(d\underline{\eta} \mid \underline{\eta}')\, \nu(d\underline{\eta}') - P(d\underline{\eta}' \mid \underline{\eta})\, \nu(d\underline{\eta})] = 0 \tag{B.1}$$

where the integration is over the $\underline{\eta}'$ variable. The evolution P is said to be *reversible* or to satisfy *detailed balance* with respect to ν if the integrand in (B.1) vanishes identically, i.e., if

$$P(d\underline{\eta} \mid \underline{\eta}')\, \nu(d\underline{\eta}') = P(d\underline{\eta}' \mid \underline{\eta})\, \nu(d\underline{\eta}) \tag{B.2}$$

It is natural to ask when a given PCA satisfies (B.2). It turns out that a very simple criterion can be given in terms of the representation (1.3): (B.2) will be satisfied if $h(\underline{\eta})$ has the form

$$h_i(\underline{\eta}) = \tanh\left[\sum_{j \in U+i} J(i-j)\,\eta_j + b\right] \tag{B.3}$$

with symmetric inputs: $J(k) = J(-k)$. The corresponding stationary ν is then given formally (or in finite volume) by

$$\nu(d\underline{\eta}) = Z^{-1} \exp \sum_i \left[b\eta_i + \log 2 \cosh\left(\sum_j J(i-j)\,\eta_j + b\right)\right] \rho_0(d\underline{\eta}) \tag{B.4}$$

with Z^{-1} a normalization constant and with ρ_0 the Bernoulli measure

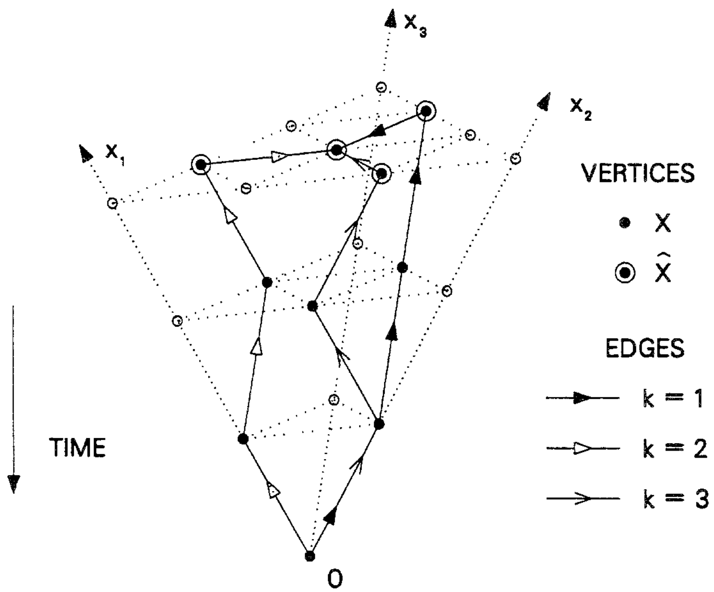

Fig. 5. Simple Toom graph in symmetric presentation. Edges of type k cannot be drawn parallel to axis k; current is conserved at each vertex.

giving all spin configurations equal weight. Thus ν is a Gibbs measure for the d-dimensional Hamiltonian $H^{(d)} = \sum_i H_i^{(d)}(\eta)$, where

$$H_i^{(d)}(\eta) = -b\eta_i - \log 2 \cosh \left(\sum_j J(i-j)\, \eta_j + b \right) = -b\eta_i - \sum_A J_A^{(d)} \eta_{A+i} \tag{B.5}$$

for appropriate constants $J_A^{(d)}$ defined for $A \subset U$. The Hamiltonian of (B.5) is of course to be distinguished from the $(d+1)$-dimensional ESM Hamiltonian defined in (1.10), which here becomes

$$H_{n,i}(\underline{\sigma}) = -2b\sigma_{n,i} - \sum_j \sigma_{n,i} J(i-j)\, \sigma_{n-1,j} - H_i^{(d)}(\underline{\sigma}_{n-1}) \tag{B.6}$$

It should be noted that the special form of the Hamiltonian $H^{(d)}$ precludes, in general, the construction of a PCA dynamics which will

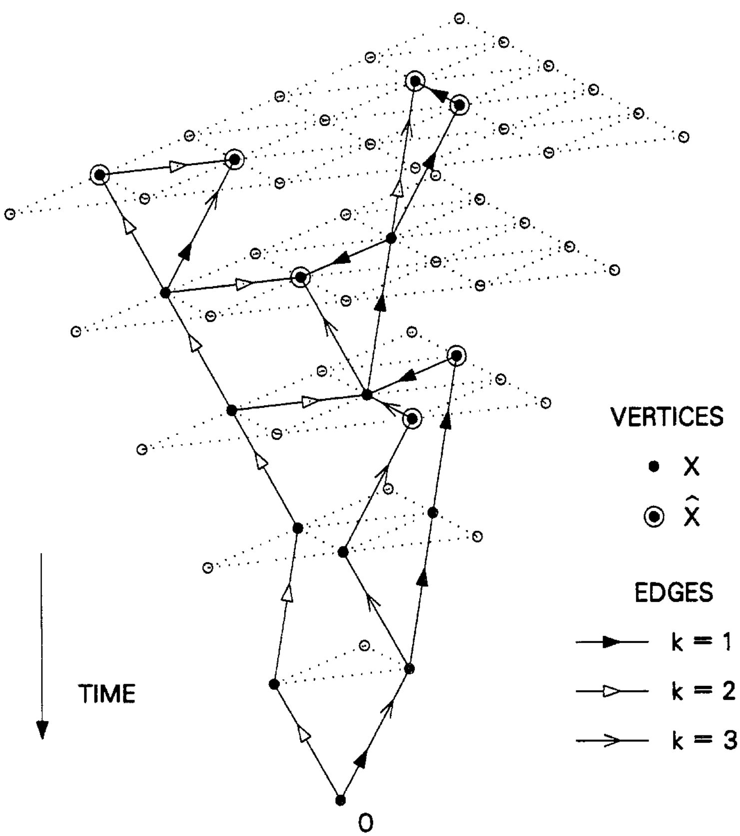

Fig. 6. Toom graph. Axes are as in Figure 5.

satisfy detailed balance for an *a priori* given Gibbs measure $\tilde{\nu}$ arising from some Hamiltonian $\tilde{H}^{(d)}$; this is in contrast to continuous time or sequential updatings where a suitable choice of stochastic dynamics is always possible. On the other hand, the construction of Domany[1] discussed in Examples 5.1 and 5.2 provides, for certain Hamiltonians with one- and two-body interactions (see (B.12)), a solution to the difficulty. Suppose that the d-dimensional lattice $\mathbb{L}^{(0)}$ may be written as $\mathbb{L}^{(0)} = \mathbb{L}^{(+)} \cup \mathbb{L}^{(-)}$; we will write $\underline{\eta}^{(\pm)}$ for the spin configurations on the separate sublattices and $\tilde{\nu}(d\underline{\eta}^{(\pm)} \mid \underline{\eta}^{(\mp)})$ for the measure on one sublattice conditioned on the spin configuration on the other. Now consider a "top-step" updating of the spin configuration on $\mathbb{L}(0)$, defined by

$$P(d\underline{\sigma}_{n+1} \mid \underline{\sigma}_n) = P^{(+)}(d\underline{\sigma}^{(+)}_{n+1} \mid \underline{\sigma}^{(+)}_n \underline{\sigma}^{(-)}_{n+1})\, P^{(-)}(d\underline{\sigma}^{(-)}_{n+1} \mid \underline{\sigma}^{(+)}_n \underline{\sigma}^{(-)}_n) \tag{B.7}$$

and the "time reversed adjoint" updating

$$P^*(d\underline{\sigma}_{n+1} \mid \underline{\sigma}_n) = P^{(-)}(d\underline{\sigma}^{(-)}_{n+1} \mid \underline{\sigma}^{(+)}_{n+1} \underline{\sigma}^{(-)}_n)\, P^{(+)}(d\underline{\sigma}^{(+)}_{n+1} \mid \underline{\sigma}^{(+)}_n \underline{\sigma}^{(-)}_n) \tag{B.8}$$

A modified detail balance condition, which ensures that the stationary stochastic process looks statistically the same when run forwards and backwards in time, is then

$$P(d\underline{\eta} \mid \underline{\eta}')\, \tilde{\nu}(d\underline{\eta}') = P^*(d\underline{\eta}' \mid \underline{\eta})\, \tilde{\nu}(d\underline{\eta}) \tag{B.9}$$

The condition (B.9) is clearly satisfied if we take

$$P^{(\pm)}(d\underline{\eta}^{(\pm)} \mid \underline{\eta}') = \tilde{\nu}(d\underline{\eta}^{(\pm)} \mid \underline{\eta}'^{(\mp)}) \tag{B.10}$$

Note that if the conditional measures are in fact product measures, then the updatings $P^{(\pm)}$ are of PCA type, but the two-step updating P is not, since the measure on $\underline{\sigma}_{n+1}$, conditioned on $\underline{\sigma}_n$, is not a product measure. This will be the case if $\tilde{\nu}$ is the Gibbs measure

$$\tilde{\nu}(\underline{\eta}) = Z^{-1} \exp[-H^{(d)}(\underline{\eta})] \tag{B.11}$$

with

$$H^{(d)}(\underline{\eta}) = \sum_{\substack{i \in \mathbb{L}^{(\mp)} \\ j \in \mathbb{L}^{(\pm)}}} J(i-j)\, \eta_i \eta_j'$$

where $J(k) = J(-k)$. A uniform magnetic field may also be included. The models of Examples 5.1 and 5.2 are of this form.

More generally, one may take

$$P^{(\pm)}(d\underline{\eta}^{(\pm)} \mid \underline{\eta}') = \prod_{i \in \mathbb{L}^{(\pm)}} \frac{\exp - \eta_i R_i^{(\pm)}(\underline{\eta}')}{2 \cosh R_i^{(\pm)}(\underline{\eta}')}\, \rho_0(d\underline{\eta}^{(\pm)}) \tag{B.13}$$

with

$$R_i^{(\pm)}(\underline{\eta}') = \sum_{j \in \mathbb{L}^{(\mp)}} J(i-j)\,\eta_j' + f_i^{(\pm)}(\underline{\eta}'^{(\mp)})\eta_i' \tag{B.14}$$

where $J(k) = J(-k)$ and $f_i^{(\pm)}(\underline{\eta}'^{(\mp)})$ vanishes whenever $\sum_{j \in \mathbb{L}^{(\mp)}} J(i-j)\eta_j' \neq 0$ and has some constant value, independent of $\underline{\eta}'$, otherwise. The resulting P of (B.7) will still satisfy detailed balance with respect to the Gibbs measure (B.11)–(B.12). This generalization of the Domany rules permits us to write the stationary measure of a one dimensional PCA which is updated alternatively, on the even and odd sites, according to a majority rule of a site and its two neighbors with noise, as the Gibbs state of a one-dimensional Ising model with nearest-neighbor interactions. It follows that the PCA will be ergodic and that the corresponding ESM will have no phase transitions. As remarked earlier, the corresponding result for simultaneous updatings has recently been proven by Gray.[16]

ACKNOWLEDGMENTS

We thank C. Bennett, J. Bricmont, E. Domany, D. Fischer, J. Fritz, S. Goldstein, L. Gray, I. Kanter, S. Shlosman, R. Schonmann, H. Spohn, and J. Steif for helpful conversations. We also thank S. A. Berezner for a careful reading of earlier versions of this article, and the detection of several errors. This work was supported in part by NSF grant DMR 86-12369.

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